

FACTORIZATION FORMULAS OF K - k -SCHUR FUNCTIONS I

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ABSTRACT. We give some new formulas about factorizations of K - k -Schur functions $g_{\lambda}^{(k)}$, analogous to the k -rectangle factorization formula $s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_{\lambda}^{(k)}$ of k -Schur functions, where λ is any k -bounded partition and R_t denotes the partition (t^{k+1-t}) called k -rectangle. Although a formula of the same form does not hold for K - k -Schur functions, we can prove that $g_{R_t}^{(k)}$ divides $g_{R_t \cup \lambda}^{(k)}$, and in fact more generally that $g_P^{(k)}$ divides $g_{P \cup \lambda}^{(k)}$ for any multiple k -rectangles $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ and any k -bounded partition λ . We give the factorization formula of such $g_P^{(k)}$ and the explicit formulas of $g_{P \cup \lambda}^{(k)} / g_P^{(k)}$ in some cases, including the case where λ is a partition with a single part as the easiest example.

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1. INTRODUCTION

Let k be a positive integer. K - k -Schur functions $g_{\lambda}^{(k)}$ are inhomogeneous symmetric functions parametrized by k -bounded partitions λ , namely by the weakly decreasing strictly positive integer sequences $\lambda = (\lambda_1, \dots, \lambda_l)$, $l \in \mathbb{Z}_{\geq 0}$, whose terms are all bounded by k . They are K -theoretic analogues of another family of symmetric functions called k -Schur functions, which are homogeneous and also parametrized by k -bounded partitions.

Historically, k -Schur functions were first introduced by Lascoux, Lapointe and Morse [LLM03], and subsequent studies led to several (conjecturally equivalent) characterizations of $s_{\lambda}^{(k)}$ such as the Pieri-like formula due to Lapointe and Morse [LM07], and Lam proved that k -Schur functions correspond to the Schubert basis of homology of the affine Grassmannian [Lam08]. Moreover it was shown by Lam and Shimozono that k -Schur functions play a central role in the explicit description of the Peterson isomorphism between quantum cohomology of the Grassmannian and homology of the affine Grassmannian up to suitable localizations [LS12].

These developments have analogues in K -theory. Lam, Schilling and Shimozono [LSS10] characterized the K -theoretic k -Schur functions as the Schubert basis of the K -homology of the affine Grassmannian, and Morse [Mor12] investigated them from a combinatorial viewpoint, giving various properties including the Pieri-like formulas using affine set-valued strips (the form using cyclically decreasing words are also given in [LSS10]).

In this paper we start from this combinatorial characterization (see Definition 7) and show certain new factorization formulas of K - k -Schur functions.

Among the k -bounded partitions, those of the form $(t^{k+1-t}) = \underbrace{(t, \dots, t)}_{k+1-t} =: R_t$,

$1 \leq t \leq k$, called k -rectangle, play a special role. In particular, if a k -bounded partition has the form $R_t \cup \lambda$, where the symbol \cup denotes the operation of concatenating the two sequences and reordering the terms in the weakly decreasing order, then the corresponding k -Schur function has the following factorization property [LM07, Theorem 40]:

$$(1) \quad s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_{\lambda}^{(k)}.$$

Note that, under the bijection between the set of k -bounded partitions and the set of affine Grassmannian elements in the affine symmetric group, the correspondent of the k -rectangle R_i is congruent, in the extended affine Weyl group, to the translation $t_{-\varpi_i^\vee}$ by the negative of a fundamental coweight, modulo left multiplication by the length zero elements.

It is suggested in [LSS10, Remark 7.4] that the K - k -Schur functions should also possess similar properties, including the divisibility of $g_{R_t \cup \lambda}^{(k)}$ by $g_{R_t}^{(k)}$. The present work is an attempt to materialize this suggestion.

We do show in Proposition 14 that $g_{R_t}^{(k)}$ divides $g_{R_t \cup \lambda}^{(k)}$ in the ring $\Lambda^{(k)} = \mathbb{Z}[h_1, \dots, h_k]$, where h_i denotes the complete homogeneous symmetric functions of degree i , of which the K - k -Schur functions form a basis. However, unlike the case of k -Schur

functions, the quotient $g_{R_t \cup \lambda}^{(k)} / g_{R_t}^{(k)}$ is not a single term $g_\lambda^{(k)}$ but, in general, a linear combination of K - k -Schur functions with leading term $g_\lambda^{(k)}$, namely in which $g_\lambda^{(k)}$ is the only highest degree term. Even the simplest case where λ consists of a single part (r) , $1 \leq r \leq k$, displays this phenomenon: we show in Theorem 23 that

$$(2) \quad g_{R_t \cup (r)}^{(k)} = \begin{cases} g_{R_t}^{(k)} \cdot g_{(r)}^{(k)} & (\text{if } t < r), \\ g_{R_t}^{(k)} \cdot (g_{(r)}^{(k)} + g_{(r-1)}^{(k)} + \cdots + g_{\emptyset}^{(k)}) & (\text{if } t \geq r) \end{cases}$$

(actually we have $g_{(s)}^{(k)} = h_s$ for $1 \leq s \leq k$, and $g_{\emptyset}^{(k)} = h_0 = 1$). So we may ask:

Question 1. Which $g_\mu^{(k)}$, besides $g_\lambda^{(k)}$, appear in the quotient $g_{R_t \cup \lambda}^{(k)} / g_{R_t}^{(k)}$? With what coefficients?

A k -bounded partition can always be written in the form $R_{t_1} \cup \cdots \cup R_{t_m} \cup \lambda$ with λ not having so many repetitions of any part as to form a k -rectangle. In such an expression we temporarily call λ the remainder, although this term is only used in the Introduction. Proceeding in the direction of Question 1, one ultimate goal may be to give a factorization formula in terms of the k -rectangles and the remainder. In the case of k -Schur functions, the straightforward factorization in (1) above leads to the formula $s_{R_{t_1} \cup \cdots \cup R_{t_m} \cup \lambda}^{(k)} = s_{R_{t_1}}^{(k)} \cdots s_{R_{t_m}}^{(k)} s_\lambda^{(k)}$. On the contrary, with K - k -Schur functions, the simplest case having a multiple k -rectangle, to be shown in the author's following paper [Taka], gives

$$(3) \quad g_{R_t \cup R_t}^{(k)} = g_{R_t}^{(k)} \sum_{\lambda \subset R_t} g_\lambda^{(k)}.$$

Hence we cannot expect $g_{R_t \cup R_t}^{(k)}$ to be divisible by $g_{R_t}^{(k)}$ twice. Instead, upon organizing the part consisting of k -rectangles in the form $R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m}$ with $t_1 < \cdots < t_m$ and $a_i \geq 1$ ($1 \leq i \leq m$), with $R_t^a = \underbrace{R_t \cup \cdots \cup R_t}_a$, actually we show in Proposition

14 that

$$g_{R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m} \cup \lambda}^{(k)} \text{ is divisible by } g_{R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m}}^{(k)},$$

which actually holds whether or not λ is the remainder. Then we can subdivide our goal as follows:

Question 1'. Which $g_\mu^{(k)}$, besides $g_\lambda^{(k)}$, appear in the quotient $g_{P \cup \lambda}^{(k)} / g_P^{(k)}$ where $P = R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m}$, and with what coefficients?

Question 2. How can $g_{R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m}}^{(k)}$ be factorized?

In this (and author's following) paper, we give a reasonably complete answer to Question 2, and partial answers to Question 1'. For Question 2, we first show in Theorem 31 that multiple k -rectangles of different sizes entirely split, namely that we have $g_{R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m}}^{(k)} = g_{R_{t_1}^{a_1}}^{(k)} \cdots g_{R_{t_m}^{a_m}}^{(k)}$. Then in the following paper [Taka], we show that for each $1 \leq t \leq k$ and $a > 1$, we have a nice factorization $g_{R_t^a}^{(k)} =$

$g_{R_t}^{(k)} \left(\sum_{\lambda \subset R_t} g_{\lambda}^{(k)} \right)^{a-1}$, generalizing the formula (3). Thus, we have

$$g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}}^{(k)} = g_{R_{t_1}}^{(k)} \left(\sum_{\lambda^{(1)} \subset R_{t_1}} g_{\lambda^{(1)}}^{(k)} \right)^{a_1-1} \cdots g_{R_{t_m}}^{(k)} \left(\sum_{\lambda^{(m)} \subset R_{t_m}} g_{\lambda^{(m)}}^{(k)} \right)^{a_m-1}.$$

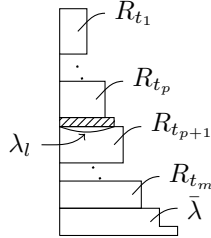
For Question 1', unfortunately we cannot give a complete answer yet. Still we obtain some nice explicit formulas, including the case (2).

We first show an auxiliary result that, being given $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ and putting $Q = R_{t_1} \cup \dots \cup R_{t_m}$ without multiplicities, we have $g_{P \cup \lambda}^{(k)} / g_P^{(k)} = g_{Q \cup \lambda}^{(k)} / g_Q^{(k)}$ for any λ . Thus we can reduce Question 1' to the case where the k -rectangles are of all different sizes.

We then derive explicit formulas in some limited cases where, writing $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\bar{\lambda} = (\lambda_1, \dots, \lambda_{l-1})$, the parts of λ except for λ_l are all larger than the widths of the k -rectangles, and $\bar{\lambda}$ is contained in a k -rectangle.

An easiest case is where $\lambda = (r)$ consists of a single part, which generalizes the case (2). Namely we show that if $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ with $t_1 < \dots < t_m$ and $a_1, \dots, a_m > 0$ and $0 \leq r \leq k$, then $g_{P \cup (r)}^{(k)}$ decomposes as $g_P^{(k)} \cdot \sum_{s=0}^r \binom{\alpha_P(r)+s-1}{s} g_{(r-s)}^{(k)}$, where $\alpha_P(u) = \#\{i \mid 1 \leq i \leq m, t_i \geq u\}$. Considering the case $m = 1$ and $a_1 = 1$, we obtain the formula (2).

Generalizing this case, we show in Theorem 30 that if P and $\alpha_P(u)$ are the same as above and $\lambda = (\lambda_1, \dots, \lambda_l)$ satisfies $\lambda_{l-1} > t_m$ and $\bar{\lambda} = (\lambda_1, \dots, \lambda_{l-1})$ is contained in a k -rectangle, then $g_{P \cup \lambda}^{(k)}$ decomposes as $g_P^{(k)} \cdot \sum_{s=0}^{\lambda_l} \binom{\alpha_P(\lambda_l)+s-1}{s} g_{\bar{\lambda} \cup (\lambda_l-s)}^{(k)}$. In particular, if $t_n < \lambda_l$, the summation on the right-hand side consists of a single term $g_{\lambda}^{(k)}$.



In this figure $p = m - \alpha_P(\lambda_l)$ and $a_i = 1$ for all i .

It is worth noting that, in all cases we have seen, $g_{P \cup \lambda}^{(k)} / g_P^{(k)}$ is a linear combination of K - k -Schur functions with *positive coefficients*. Moreover, if $P = R_t$, it seems that each coefficient is 0 or 1 and the set of μ such that the coefficient of $g_{\mu}^{(k)}$ in $g_{P \cup \lambda}^{(k)} / g_P^{(k)}$ is 1 is an *interval* (with respect to the strong order. See Conjecture 33). Anyway, it should be interesting to study the geometric meaning of these results and conjectures.

This paper is organized as follows. In Section 2, we review some basic notations and facts about combinatorial backgrounds of K - k -Schur functions. In Section 3, we show some auxiliary results which provide a basis for our work. In Section 4, we give explicit factorization formulas in an easiest case where the remainder consists of a single part. In Section 5, we generalize the result of the previous section and

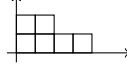
give a “straightforward factorization” formula for a multiple k -rectangles of different sizes. In Section 6, we state some observations and conjectures.

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2. PRELIMINARIES

In this section we review some requisite combinatorial backgrounds. For detailed definitions, see for instance [LLM⁺14, Chapter 2] or [Mac95, Chapter I].

2.1. Partitions. Let \mathcal{P} denote the set of partitions. A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots) \in \mathcal{P}$ is identified with its *Young diagram* (or *shape*), for which we use the French notation here. quadrant¹ of Cartesian plane so that there are λ_i boxes arranged in left justified way in the i -th row from the bottom.

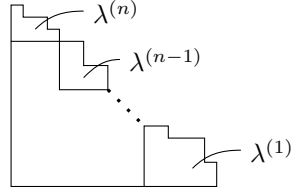


the Young diagram of $(4, 2)$

We denote the *size* of a partition λ by $|\lambda|$, the *length* by $l(\lambda)$, and the *conjugate* by λ' . For partitions λ, μ we say $\lambda \subset \mu$ if $\lambda_i \leq \mu_i$ for all i . The *dominance order* \leq on \mathcal{P} is defined by saying that $\lambda \leq \mu$ if $|\lambda| = |\mu|$ and $\sum_{i=1}^r \lambda_i \leq \sum_{i=1}^r \mu_i$ for all $r \geq 1$. Sometimes we abbreviate *horizontal strip* (resp. *vertical strip*) (of size r) to (r) -h.s. (resp. (r) -v.s.). For a partition λ and a cell $c = (i, j)$ in λ , we denote the *hook length* of c in λ by $\text{hook}_c(\lambda) = \lambda_i + \lambda'_j - i - j + 1$.

For a partition λ , a *removable corner* of λ (or λ -removable corner) is a cell $(i, j) \in \lambda$ with $(i, j + 1), (i + 1, j) \notin \lambda$. $(i, j) \in (\mathbb{Z}_{>0})^2 \setminus \lambda$ is said to be an *addable corner* of λ (or λ -addable corner) if $(i, j - 1), (i - 1, j) \in \lambda$ with the understanding that $(0, j), (j, 0) \in \lambda$. In order to avoid making equations too wide, we may denote removable corner (resp. addable corner) briefly by rem. cor. (resp. add. cor.). A cell $(i, j) \in \lambda$ is called *extremal* if $(i + 1, j + 1) \notin \lambda$.

For partitions $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$, $\mu = (\mu_1, \dots, \mu_{l(\mu)})$, we write $\lambda \oplus \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_1, \dots, \lambda_{l(\lambda)} + \mu_1, \mu_1, \dots, \mu_{l(\mu)})$. For partitions $\lambda^{(1)}, \dots, \lambda^{(n)}$, we define $\lambda^{(1)} \oplus \dots \oplus \lambda^{(n)} = (\lambda^{(1)} \oplus \dots \oplus \lambda^{(n-1)}) \oplus \lambda^{(n)}$, recursively.



the shape of $\lambda^{(1)} \oplus \dots \oplus \lambda^{(n)}$

¹We use the French notation

2.2. Bounded partitions, cores, affine Grassmannian elements, and k -rectangles R_t . A partition λ is called k -bounded if $\lambda_1 \leq k$. Let \mathcal{P}_k be the set of all k -bounded partitions. An r -core (or simply a *core* if no confusion can arise) is a partition none of whose cells have a hook length equal to r . We denote by \mathcal{C}_r the set of all r -core partitions. When we consider a partition as a core, the notion of size differs from the usual one: the *length* (or *size*) of an r -core κ is the number of cells in κ whose hook length is smaller than r , and denoted by $|\kappa|_r$.

The *affine symmetric group* \tilde{S}_{k+1} is given by generators $\{s_0, s_1, \dots, s_k\}$ and relations $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ for $i - j \not\equiv 0, 1, k \pmod{k+1}$, with all indices are considered mod $(k+1)$. Note that the symmetric group S_{k+1} generated by $\{s_1, \dots, s_k\}$ is a subgroup of \tilde{S}_{k+1} . We identify the left cosets of \tilde{S}_{k+1}/S_{k+1} with their minimal length representatives, which we call *affine Grassmannian elements*. Namely, the set of affine Grassmannian elements is $\{w \in \tilde{S}_{k+1} \mid l(ws_i) > l(w) \ (\forall i \neq 0)\}$.

Hereafter we fix a positive integer k .

For a cell $c = (i, j)$, the *content* of c is $j - i$ and the *residue* of c is $\text{res}(c) = j - i \pmod{k+1} \in \mathbb{Z}/(k+1)$. For a set X of cells, we write $\text{Res}(X) = \{\text{res}(c) \mid c \in X\}$. We will write a λ -removable corner of residue i simply a λ -removable i -corner. For simplicity of notation, we may use an integer to represent a residue, omitting “mod $(k+1)$ ”.

We denote by R_t the partition $(t^{k+1-t}) = (t, t, \dots, t) \in \mathcal{P}_k$ for $1 \leq t \leq k$, which is called a k -rectangle. Naturally a k -rectangle is a $(k+1)$ -core.

Now we recall the bijection between the k -bounded partitions in \mathcal{P}_k , the $(k+1)$ -cores in \mathcal{C}_{k+1} , and the affine Grassmannian elements in \tilde{S}_{k+1}/S_{k+1} :

$$\begin{array}{ccc}
 \mathcal{P}_k & \begin{array}{c} \xleftarrow{\mathfrak{c}} \\ \xrightarrow{\mathfrak{p}} \end{array} & \mathcal{C}_{k+1} \\
 & \searrow \text{by taking} & \nearrow \mathfrak{s} \\
 & \tilde{S}_{k+1}/S_{k+1} &
 \end{array}$$

“word”

The maps \mathfrak{p} and \mathfrak{c} :

The map $\mathfrak{p}: \mathcal{C}_{k+1} \rightarrow \mathcal{P}_k; \kappa \mapsto \lambda$ is defined by $\lambda_i = \#\{j \mid (i, j) \in \kappa, \text{hook}_{(i,j)}(\kappa) \leq k\}$.

The map $\mathfrak{c}: \mathcal{P}_k \rightarrow \mathcal{C}_{k+1}; \lambda \mapsto \kappa$ is defined by the following procedure: given a k -bounded partition λ then work from the smallest part to the largest. For each row, calculate the hook lengths of all its cells. If there is a cell with hook length greater than k , slide this row to the right until all its cells have hook length not greater than k . In the end this process produces a skew shape μ/ν , where in fact μ is a $(k+1)$ -core. Then let κ be this μ .

Then in fact \mathfrak{p} and \mathfrak{c} are bijective and $\mathfrak{p} = \mathfrak{c}^{-1}$. See [LM05, Theorem 7] for the proof. The next lemma gives a more explicit description for \mathfrak{c} , which follows from the argument given just before [LLM⁺14, Example 1.23]:

Lemma 1. For $\lambda \in \mathcal{P}_k$ and $j \geq 1$, $\mathfrak{c}(\lambda)_j = \mathfrak{c}(\lambda)_{j+k+1-\lambda_j} + \lambda_j$.

Note that if λ is contained in a k -rectangle then $\lambda \in \mathcal{P}_k$ and $\lambda \in \mathcal{C}_{k+1}$, and besides $\mathfrak{p}(\lambda) = \lambda = \mathfrak{c}(\lambda)$.

The map \mathfrak{s} and the inverse: For $\kappa \in \mathcal{C}_{k+1}$ and $i = 0, 1, \dots, k$, we define $s_i \cdot \kappa$ as follows:

- if there is a κ -addable i -corner, then let $s_i \cdot \kappa$ be κ with all κ -addable i -corners added,
- if there is a κ -removable i -corner, then let $s_i \cdot \kappa$ be κ with all κ -removable i -corners removed,
- otherwise, let $s_i \cdot \kappa$ be κ .

In fact first and second case never occur simultaneously and $s_i \cdot \kappa \in \mathcal{C}_{k+1}$ and then we have a well-defined \tilde{S}_{k+1} -action on \mathcal{C}_{k+1} and it induces a bijection

$$\mathfrak{s} : \tilde{S}_{k+1}/S_{k+1} \longrightarrow \mathcal{C}_{k+1}; w \mapsto w \cdot \emptyset.$$

The inverse map is given by

$$\mathcal{C}_{k+1} \longrightarrow \mathcal{P}_k \longrightarrow \tilde{S}_{k+1}/S_{k+1}; \kappa \mapsto \mathfrak{p}(\kappa) \mapsto w_{\mathfrak{p}(\kappa)},$$

where w_λ is the affine permutation $s_{i_1} s_{i_2} \dots s_{i_l}$, where (i_1, i_2, \dots, i_l) is the sequence obtained by reading the residues of the cells in λ , from the shortest row to the largest, and within each row from right to left. See [LM05, Corollary 48] for the proof.

2.3. Weak order and weak strips. In this subsection we review the weak order on $\mathcal{P}_k \simeq \mathcal{C}_{k+1} \simeq \tilde{S}_{k+1}/S_{k+1}$.

For a k -bounded partition λ , its k -conjugate λ^{ω_k} is also a k -bounded partition given by $\lambda^{\omega_k} = \mathfrak{p}(\mathfrak{c}(\lambda)')$.

Definition-Proposition 2. The weak order \prec on \tilde{S}_{k+1}/S_{k+1} is defined by the following covering relation:

$$(4) \quad w \prec v : \iff \exists i \text{ such that } s_i w = v, l(w) + 1 = l(v).$$

It is transferred to \mathcal{P}_k and \mathcal{C}_{k+1} by the bijection described above as follows:

$$(5) \quad \text{on } \mathcal{P}_k: \quad \lambda \prec \mu \iff \lambda \subset \mu, \lambda^{\omega_k} \subset \mu^{\omega_k}, |\lambda| + 1 = |\mu|.$$

$$(6) \quad \text{on } \mathcal{C}_{k+1}: \quad \tau \prec \kappa \iff \exists i \text{ such that } s_i \tau = \kappa, |\tau|_{k+1} + 1 = |\kappa|_{k+1}.$$

Proof. (4) \iff (6): see [Las01]. (5) \iff (6): see [LM05, Corollary 25]. \square

Definition-Proposition 3. For $(k+1)$ -cores $\tau \subset \kappa \in \mathcal{C}_{k+1}$, κ/τ is called a weak strip of size r (or a weak r -strip) if the following equivalent conditions hold:

- (1) κ/τ is horizontal strip and $\tau \prec \exists \tau^{(1)} \prec \dots \prec \exists \tau^{(r)} = \kappa$.
- (2) κ/τ is horizontal strip and $|\kappa|_{k+1} = |\tau|_{k+1} + r$ and $\#\text{Res}(\kappa/\tau) = r$.
- (3) $\mathfrak{p}(\kappa)/\mathfrak{p}(\tau)$ is a horizontal strip and $\mathfrak{p}(\kappa')/\mathfrak{p}(\tau')$ is a vertical strip and $|\kappa|_{k+1} = |\tau|_{k+1} + r$.
- (4) $\kappa = s_{i_1} \dots s_{i_r} \tau$ for some cyclically decreasing element $s_{i_1} \dots s_{i_r}$.
(Here, an affine permutation $w = s_{i_1} \dots s_{i_r}$ (a reduced expression) is called cyclically decreasing if i_1, \dots, i_r are distinct and j never precedes $j+1$ (taken modulo $k+1$) in the sequence $i_1 i_2 \dots i_r$. This definition is in fact independent of which reduced expression we choose.)

Proof. (1) \implies (3): see [LM05, Theorem 58]. (3) \implies (1): see [LM05, Proposition 54, Theorem 56]. (3) \implies (2): see [LM05, Theorem 56].

(4) \implies (1), (1) \implies (4): see Appendix B.

(2) \implies (1): omitted since (2) is not used in this paper. \square

2.4. Symmetric functions. Let $\Lambda = \mathbb{Z}[h_1, h_2, \dots]$ be the ring of symmetric functions, generated by the *complete symmetric functions* $h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} \dots x_{i_r}$. For a partition λ we set $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_{l(\lambda)}}$. Then $\{h_\lambda\}_{\lambda \in \mathcal{P}}$ forms a \mathbb{Z} -basis of Λ . The *Schur functions* $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ are the family of symmetric functions satisfying the *Pieri rule*:

$$h_r s_\lambda = \sum_{\mu/\lambda: \text{horizontal } r\text{-strip}} s_\mu.$$

Note that $h_r s_\lambda = s_{\lambda \cup (r)} + \sum_{\mu \triangleright \lambda \cup (r)} a_\mu s_\mu$ for some a_μ . Using this repeatedly, we can write $h_\lambda = s_\lambda + \sum_{\mu \triangleleft \lambda} K_{\mu\lambda} s_\mu$ for some coefficients $K_{\mu\lambda}$. Thus Schur functions $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ form a basis of Λ since the transformation matrix between $\{s_\lambda\}_\lambda$ and $\{h_\mu\}_\mu$ is unitriangular.

2.5. k -Schur functions. We recall a characterization of k -Schur functions given in [LM07], since it is a model for and has a relationship with K - k -Schur functions.

Definition 4 (k -Schur function via “weak Pieri rule”). k -Schur functions $\{s_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ are the family of symmetric functions such that

$$\begin{aligned} s_\emptyset^{(k)} &= 1, \\ h_r s_\lambda^{(k)} &= \sum_{\mu} s_\mu^{(k)} \quad \text{for } r \leq k \text{ and } \mu \in \mathcal{P}_k, \end{aligned}$$

summed over $\mu \in \mathcal{P}_k$ such that $\mathbf{c}(\mu)/\mathbf{c}(\lambda)$ is a weak strip of size r .

According to the fact that if $\mathbf{c}(\nu)/\mathbf{c}(\eta)$ is a weak strip then ν/η is a horizontal strip, we can write $h_\lambda = s_\lambda^{(k)} + \sum_{\mu \triangleleft \lambda} K_{\mu\lambda}^{(k)} s_\mu^{(k)}$ for $\lambda \in \mathcal{P}_k$ by the same argument as the case of Schur functions, which ensures the well-definedness of $s_\lambda^{(k)}$ and shows that $\{s_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ forms a basis of $\Lambda^{(k)} = \mathbb{Z}[h_1, \dots, h_k] \subset \Lambda$. In addition $s_\lambda^{(k)}$ is homogeneous of degree $|\lambda|$.

Note that $s_{(r)}^{(k)} = h_r$ for $1 \leq r \leq k$ since $\mathbf{c}(\lambda)/\emptyset$ is a weak r -strip if and only if $\lambda = (r)$. In [LM07, Property 39] it is proved that if $\lambda_1 + l(\lambda) \leq k + 1$ (in other words $\lambda \subset R_t$ for some t) then $s_\lambda^{(k)} = s_\lambda$.

It is proved in [LM07, Theorem 40] that

Proposition 5 (k -rectangle property). For $1 \leq t \leq k$ and $\lambda \in \mathcal{P}_k$, we have $s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_\lambda^{(k)} (= s_{R_t} s_\lambda^{(k)})$.

2.6. K - k -Schur functions $g_\lambda^{(k)}$. In [Mor12] a combinatorial characterization of K - k -Schur functions is given via an analogue of the Pieri rule, using some kind of strips called *affine set-valued strips*.

For a partition λ , $(i, j) \in (\mathbb{Z}_{>0})^2$ is called λ -blocked if $(i + 1, j) \in \lambda$.

Definition 6 (affine set-valued strip). For $r \leq k$, $(\gamma/\beta, \rho)$ is called an *affine set-valued strip of size r* (or an *affine set-valued r -strip*) if ρ is a partition and $\beta \subset \gamma$ are cores both containing ρ such that

- (1) γ/β is a weak $(r - m)$ -strip where we put $m = \#\text{Res}(\beta/\rho)$,
- (2) β/ρ is a subset of β -removable corners,
- (3) γ/ρ is a horizontal strip,
- (4) For all $i \in \text{Res}(\beta/\rho)$, all β -removable i -corners which are not γ -blocked are in β/ρ .

In this paper we employ the following characterization [Mor12, Theorem 48] of the K - k -Schur function as its definition.

Definition 7 (K - k -Schur function via an “affine set-valued” Pieri rule). *K - k -Schur functions $\{g_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ are the family of symmetric functions such that $g_\emptyset^{(k)} = 1$ and for $\lambda \in \mathcal{P}_k$ and $0 \leq r \leq k$,*

$$h_r \cdot g_\lambda^{(k)} = \sum_{(\mu, \rho)} (-1)^{|\lambda|+r-|\mu|} g_\mu^{(k)},$$

summed over (μ, ρ) such that $(\mathbf{c}(\mu)/\mathbf{c}(\lambda), \rho)$ is an affine set-valued strip of size r .

Notice that, given a weak strip γ/β , taking a ρ such that $(\gamma/\beta, \rho)$ becomes an affine set-valued strip is equivalent to choosing a subset of the set of residues $i \in \mathbb{Z}/(k+1)$ where there is at least one γ -nonblocked β -removable i -corner.

Now we introduce a notation for the convinience:

Definition 8. For partitions λ, μ , we denote by $r_{\lambda\mu}$ the number of distinct residues of λ -nonblocked μ -removable corners.

Then for a fixed weak $(r-m)$ -strip γ/β , the number of ρ such that $(\gamma/\beta, \rho)$ is an affine set-valued r -strip is equal to $\binom{r_{\gamma\beta}}{m}$. Notice that γ/β with all γ -nonblocked β -removable corners added is a horizontal strip. Therefore we can rewrite Definition 7:

Proposition 9. For $\lambda \in \mathcal{P}_k$ and $0 \leq r \leq k$,

$$(7) \quad h_r \cdot g_\lambda^{(k)} = \sum_{s=0}^r (-1)^{r-s} \sum_{\substack{\mu \\ \mathbf{c}(\mu)/\mathbf{c}(\lambda): \text{weak } s\text{-strip}}} \binom{r_{\mathbf{c}(\mu)\mathbf{c}(\lambda)}}{r-s} g_\mu^{(k)}.$$

We can prove similarly that $g_\lambda^{(k)}$ is uniquely determined by (7): for $\lambda \in \mathcal{P}_k$ and $1 \leq r \leq k$, we have $h_r g_\lambda^{(k)} = g_{\lambda \cup (r)}^{(k)} + \sum_{\mu} a_\mu g_\mu^{(k)}$ with $\mu \in \mathcal{P}_k$ satisfying $|\mu| < |\lambda \cup (r)|$ or $\mu \triangleright \lambda \cup (r)$. Thus, for $\lambda \in \mathcal{P}_k$ we can write $h_\lambda = g_\lambda^{(k)} + \sum_{\mu} \mathcal{K}_{\mu\lambda}^{(k)} g_\mu^{(k)}$, summed over $\mu \in \mathcal{P}_k$ satisfying $|\mu| < |\lambda|$ or $\mu \triangleright \lambda$. Hence $g_\lambda^{(k)}$ is well-defined and $\{g_\lambda^{(k)}\}_{\lambda \in \mathcal{P}_k}$ forms a basis of $\Lambda^{(k)}$.

Note that $g_{(r)}^{(k)} = h_r$ for $1 \leq r \leq k$ since if $(\mathbf{c}(\mu)/\emptyset, \rho)$ is an affine set-valued r -strip then $(\mu, \rho) = ((r), \emptyset)$. Moreover, though $g_\lambda^{(k)}$ is an inhomogeneous symmetric function in general, from the form of (7) we can deduce that the degree of $g_\lambda^{(k)}$ is $|\lambda|$ and its homogeneous part of highest degree is equal to $s_\lambda^{(k)}$ by using induction.

2.7. Some properties of bounded partitions and cores. In this section we review some properties which show that the k -rectangles $R_t = (t^{k+1-t})$ are important and thus it can be expected that there are some good properties of $g_\lambda^{(k)}$'s where λ can be written in the form $\lambda = R_t \cup \mu$.

Recall the weak order \prec of Definition 2.

Corollary 10. For $\mu, \lambda \in \mathcal{P}_k$ and $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ ($1 \leq t_1 < \dots < t_m \leq k$ and $a_1, \dots, a_m \in \mathbb{Z}_{>0}$),

$$\lambda \cup P \preceq \mu \iff \exists \nu \in \mathcal{P}_k, \begin{cases} \mu = \nu \cup P, \\ \lambda \preceq \nu. \end{cases}$$

Proof. See [LM04, Theorem 20] for the case where $m = 1$ and $a_1 = 1$ (i.e. $P = R_{t_1}$). The general case follows by using this case repeatedly. \square

Proposition 11. *For $\nu, \lambda \in \mathcal{P}_k$ and $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ ($1 \leq t_1 < \dots < t_m \leq k$ and $a_1, \dots, a_m \in \mathbb{Z}_{>0}$),*

$$\mathbf{c}(\nu)/\mathbf{c}(\lambda) \text{ is a weak strip} \iff \mathbf{c}(\nu \cup P)/\mathbf{c}(\lambda \cup P) \text{ is a weak strip}$$

Proof. The case where $m = 1$ and $a_1 = 1$ (i.e. $P = R_{t_1}$) is proved in the proof of [LM07, Theorem 40]. The general case follows by using this case repeatedly. \square

Corollary 12. *For $\eta, \lambda \in \mathcal{P}_k$ and $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ ($1 \leq t_1 < \dots < t_m \leq k$ and $a_1, \dots, a_m \in \mathbb{Z}_{>0}$),*

$$\mathbf{c}(\mu)/\mathbf{c}(\lambda \cup P) \text{ is a weak strip} \iff \exists \nu \in \mathcal{P}_k, \begin{cases} \mu = \nu \cup P, \\ \mathbf{c}(\nu)/\mathbf{c}(\lambda) \text{ is a weak strip.} \end{cases}$$

Proof. \implies : We have $\lambda \cup P \leq \mu$ by the definition of weak strips. Thus we can write $\mu = \exists \nu \cup P$ by Proposition 10. Then we have that $\mathbf{c}(\nu)/\mathbf{c}(\lambda)$ is a weak strip by Proposition 11.

\implies : By Proposition 11. \square

Lemma 13. *Let $\lambda \in \mathcal{P}_k$, $1 \leq t \leq k$, and let $r \in \mathbb{Z}_{\geq 0}$ such that $\lambda_r \geq t \geq \lambda_{r+1}$, where we regard $\lambda_0 = \infty$. Put $\tilde{\lambda} = \lambda \cup R_t$. Then*

$$\mathbf{c}(\tilde{\lambda})_i = \begin{cases} \mathbf{c}(\lambda)_i + t & (\text{if } i \leq r + (k + 1 - t)), \\ \mathbf{c}(\lambda)_{i-(k+1-t)} & (\text{if } i \geq (r + 1) + (k + 1 - t)). \end{cases}$$

Proof. The latter case is obvious since $\tilde{\lambda}_{i+(k+1-t)} = \lambda_i$ for $i \geq r + 1$.

For $i = r + (k + 1 - t), \dots, r + 1$,

$$\begin{aligned} \mathbf{c}(\tilde{\lambda})_i &= \mathbf{c}(\tilde{\lambda})_{i+k+1-\tilde{\lambda}_i} + \tilde{\lambda}_i && (\text{by Lemma 1}) \\ &= \mathbf{c}(\tilde{\lambda})_{i+k+1-t} + t && (\text{since } \tilde{\lambda}_i = t) \\ &= \mathbf{c}(\lambda)_i + t. && (\text{by the latter case}) \end{aligned}$$

Then for $i = r, r - 1, \dots, 1$, by descending induction on i ,

$$\begin{aligned} \mathbf{c}(\tilde{\lambda})_i &= \mathbf{c}(\tilde{\lambda})_{i+k+1-\tilde{\lambda}_i} + \tilde{\lambda}_i && (\text{by Lemma 1}) \\ &= \mathbf{c}(\tilde{\lambda})_{\underbrace{i+k+1-\lambda_i}_{\leq r+k+1-t}} + \lambda_i && (\text{since } i \leq r) \\ &= \mathbf{c}(\lambda)_{i+k+1-\lambda_i} + t + \lambda_i && (\text{induction hypothesis}) \\ &= \mathbf{c}(\lambda)_i + t. && (\text{by Lemma 1}) \end{aligned}$$

\square

Remark. There are more than one candidates for r if λ has a part equal to t , thus in such situations both equalities of the above lemma may hold for some i .

3. POSSIBILITY OF FACTORING OUT $g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}}^{(k)}$ AND SOME OTHER GENERAL RESULTS

Recall how to prove the formula $s_{R_t \cup \lambda}^{(k)} = s_{R_t}^{(k)} s_{\lambda}^{(k)}$ in [LM07]: first consider a linear map Θ extending $s_{\lambda}^{(k)} \mapsto s_{R_t \cup \lambda}^{(k)}$ for all $\lambda \in \mathcal{P}_k$. Then from the weak Pieri rule it was shown that it commutes with the multiplication by h_r , and thus that Θ coincides with the multiplication by $s_{R_t}^{(k)}$. In the case of K - k -Schur functions, a similar map Θ does not commute with the multiplication of h_r since the Pieri rule is different in lower terms. However, it holds that $g_{R_t}^{(k)}$ divides $g_{R_t \cup \lambda}^{(k)}$. We prove it in a slightly more general form.

The following notation is often referred later:

(NP) Let $1 \leq t_1, \dots, t_m \leq k$ be distinct integers and $a_i \in \mathbb{Z}_{>0}$ ($1 \leq i \leq m$), where $m \in \mathbb{Z}_{>0}$. Then we put

$$P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m},$$

$$\alpha_P(u) = \#\{t_v \mid 1 \leq v \leq m, t_v \geq u\} \quad \text{for each } u \in \mathbb{Z}_{>0}.$$

Proposition 14. *Let P be as in the above (NP). Then, for $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{P}_k$, we have $g_P^{(k)} \mid g_{\lambda \cup P}^{(k)}$ in the ring $\Lambda^{(k)}$.*

Remark. Note that λ may still have the form $\lambda = R_t \cup \mu$. Hereafter we will not repeat the same remark in similar statements.

Proof. we prove it by induction on λ , with respect to the order \leq defined by $\mu \leq \lambda \iff |\mu| < |\lambda|$ or $(|\mu| = |\lambda| \text{ and } \mu \triangleright \lambda)$. The statement is obvious when $\lambda = \emptyset$.

Assume $\lambda \neq \emptyset$ and put $\hat{\lambda} = (\lambda_1, \dots, \lambda_{l-1})$. Then

$$g_{P \cup \hat{\lambda}}^{(k)} \cdot g_{(\lambda_l)}^{(k)} = g_{P \cup \lambda}^{(k)} + \sum_{\mu} a_{\lambda \mu} g_{P \cup \mu}^{(k)}$$

for some coefficients $a_{\lambda \mu}$, since adding a weak strip to $P \cup \hat{\lambda}$ yields a k -bounded partition in the form of $P \cup \mu$ for some $\mu \in \mathcal{P}_k$, by Proposition 10. Here μ in the summation runs under the condition $|\mu| < |\lambda|$ or $\mu \triangleright \lambda$. By induction hypothesis $g_{P \cup \hat{\lambda}}^{(k)}$ and $g_{P \cup \mu}^{(k)}$ are divisible by $g_P^{(k)}$ if $|\mu| < |\lambda|$ or $\mu \triangleright \lambda$. This completes the proof. \square

Since the homogeneous part of highest degree of $g_{\lambda}^{(k)}$ is equal to $s_{\lambda}^{(k)}$ for any λ , it follows from Propositions 5 and 14 that

Corollary 15. *Let P be as in (NP). Then, for any $\lambda \in \mathcal{P}_k$, we can write*

$$g_{P \cup \lambda}^{(k)} = g_P^{(k)} \left(g_{\lambda}^{(k)} + \sum_{\mu} a_{\lambda \mu} g_{\mu}^{(k)} \right),$$

summing over $\mu \in \mathcal{P}_k$ such that $|\mu| < |\lambda|$, for some coefficients $a_{\lambda \mu}$ (depending on P).

Now we are interested in finding an explicit description of $g_{P \cup \lambda}^{(k)} / g_P^{(k)}$. Let us consider the case $P = R_t$ for simplicity.

As noted above, a linear map Θ extending $g_{\lambda}^{(k)} \mapsto g_{R_t \cup \lambda}^{(k)}$ ($\forall \lambda \in \mathcal{P}_k$) does not coincide with the multiplication of $g_{R_t}^{(k)}$ because it does not commute with the multiplication by h_r in the first place.

However, in the remaining part of this section, we can prove that the restriction of Θ to the subspace spanned by $\{g_{R_t \cup \mu}^{(k)}\}_{\mu \in \mathcal{P}_k}$ (in fact this is the principal ideal generated by $g_{R_t}^{(k)}$) commutes with the multiplication by h_r , and thus it coincides with the multiplication of $\Theta(g_{R_t}^{(k)})/g_{R_t}^{(k)} = g_{R_t \cup R_t}^{(k)}/g_{R_t}^{(k)}$ on that ideal (Proposition 18). Thus it is of interest to describe the value of $g_{R_t \cup R_t}^{(k)}/g_{R_t}^{(k)}$, which is shown to be $\sum_{\nu \subset R_t} g_{\nu}^{(k)}$ in the author's following paper [Taka].

Now let us begin with seeing how Θ and the multiplication by h_r do *not* commute. Recall the K - k -Schur version of the Pieri rule (7)

$$h_r \cdot g_{\lambda}^{(k)} = \sum_{s=0}^r (-1)^{r-s} \sum_{\substack{\nu \\ \mathbf{c}(\nu)/\mathbf{c}(\lambda): \text{weak } s\text{-strip}}} \binom{r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)}}{r-s} g_{\nu}^{(k)},$$

and compare with the formula obtained by replacing λ with $R_t \cup \lambda$:

$$(8) \quad h_r \cdot g_{R_t \cup \lambda}^{(k)} = \sum_{s=0}^r (-1)^{r-s} \sum_{\substack{\eta \\ \mathbf{c}(\eta)/\mathbf{c}(R_t \cup \lambda): \text{weak } s\text{-strip}}} \binom{r_{\mathbf{c}(\eta)\mathbf{c}(R_t \cup \lambda)}}{r-s} g_{\eta}^{(k)}.$$

By Corollary 12, the summation in (8) is formed for all η having the form $\eta = R_t \cup \nu$ such that $\mathbf{c}(\nu)/\mathbf{c}(\lambda)$ is a weak s -strip. Hence the right-hand side of (8) differs from what is obtained by replacing each ν in the right-hand side of (7) by $R_t \cup \nu$ according to the difference between $r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)}$ and $r_{\mathbf{c}(R_t \cup \nu)\mathbf{c}(R_t \cup \lambda)}$.

The next lemma says $r_{\mathbf{c}(R_t \cup \nu)\mathbf{c}(R_t \cup \lambda)} = r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)}$ holds if λ has a part equal to t .

Lemma 16. *For $\nu, \lambda \in \mathcal{P}_k$ such that λ has a part equal to t and $\mathbf{c}(\nu)/\mathbf{c}(\lambda)$ is a weak strip, we have $r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)} = r_{\mathbf{c}(\nu \cup R_t)\mathbf{c}(\lambda \cup R_t)}$.*

Proof. We write $\tilde{\lambda} = \lambda \cup R_t$ and $\tilde{\nu} = \nu \cup R_t$. We take r such that $\lambda_r = t > \lambda_{r+1}$ (then $\nu_r \geq t = \lambda_r \geq \nu_{r+1}$ since ν/λ is a horizontal strip). Then we have

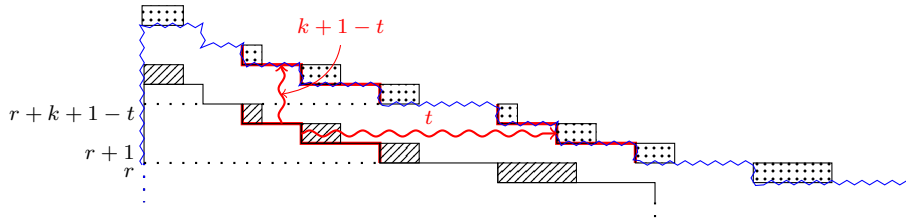
$$\begin{aligned} \tilde{\lambda}_r &= \tilde{\lambda}_{r+1} = \cdots = \tilde{\lambda}_{r+k+1-t} = t \\ \tilde{\nu}_{r+1} &= \cdots = \tilde{\nu}_{r+k+1-t} = t, \end{aligned}$$

therefore by Lemma 13

$$\begin{aligned} \mathbf{c}(\tilde{\lambda})_i &= \mathbf{c}(\lambda)_i + t & (i \leq r + (k+1-t)) \\ \mathbf{c}(\tilde{\lambda})_i &= \mathbf{c}(\lambda)_{i-(k+1-t)} & (i \geq r + (k+1-t)) \end{aligned}$$

(here we applied Lemma 13 to λ and $r-1$ for the lower equation) and

$$\begin{aligned} \mathbf{c}(\tilde{\nu})_i &= \mathbf{c}(\nu)_i + t & (i \leq r + (k+1-t)) \\ \mathbf{c}(\tilde{\nu})_i &= \mathbf{c}(\nu)_{i-(k+1-t)} & (i > r + (k+1-t)). \end{aligned}$$



$$\left(\text{Here } \begin{cases} \text{---} & : \text{outline of } \mathbf{c}(\lambda) \\ \text{---} & : \text{outline of } \mathbf{c}(\tilde{\lambda}) \\ \text{▨} & : \mathbf{c}(\nu)/\mathbf{c}(\lambda) \\ \text{▤} & : \mathbf{c}(\tilde{\nu})/\mathbf{c}(\tilde{\lambda}) \end{cases} \right)$$

Then,

- (1) if $i < r + (k + 1 - t)$,
 $(i, \mathbf{c}(\tilde{\lambda})_i)$ is a $\mathbf{c}(\tilde{\lambda})$ -removable corner $\iff (i, \mathbf{c}(\lambda)_i)$ is a $\mathbf{c}(\lambda)$ -removable corner,
- (2) if $i \geq r + (k + 1 - t)$,
 $(i, \mathbf{c}(\tilde{\lambda})_i)$ is a $\mathbf{c}(\tilde{\lambda})$ -removable corner $\iff (i - (k + 1 - t), \mathbf{c}(\lambda)_{i-(k+1-t)})$ is a $\mathbf{c}(\lambda)$ -removable corner.

Moreover, when $(i, \mathbf{c}(\tilde{\lambda})_i)$ is a $\mathbf{c}(\tilde{\lambda})$ -removable corner (of residue a), we consider two cases:

- (1) if $i < r + (k + 1 - t)$. Then

$$\begin{aligned} (i, \mathbf{c}(\tilde{\lambda})_i) \text{ is } \mathbf{c}(\tilde{\nu})\text{-blocked} &\iff \mathbf{c}(\tilde{\lambda})_i \leq \mathbf{c}(\tilde{\nu})_{i+1} \\ &\iff \mathbf{c}(\lambda)_i + t \leq \mathbf{c}(\nu)_{i+1} + t \\ &\iff (i, \mathbf{c}(\lambda)_i) \text{ is } \mathbf{c}(\nu)\text{-blocked,} \end{aligned}$$

and the residue of $(i, \mathbf{c}(\lambda)_i)$ is $a - t$.

- (2) if $i \geq r + (k + 1 - t)$. Then

$$\begin{aligned} (i, \mathbf{c}(\tilde{\lambda})_i) \text{ is } \mathbf{c}(\tilde{\nu})\text{-blocked} &\iff \mathbf{c}(\tilde{\lambda})_i \leq \mathbf{c}(\tilde{\nu})_{i+1} \\ &\iff \mathbf{c}(\lambda)_{i-(k+1-t)} \leq \mathbf{c}(\nu)_{i+1-(k+1-t)} \\ &\iff (i - (k + 1 - t), \mathbf{c}(\lambda)_{i-(k+1-t)}) \text{ is } \mathbf{c}(\nu)\text{-blocked,} \end{aligned}$$

and the residue of $(i - (k + 1 - t), \mathbf{c}(\lambda)_{i-(k+1-t)})$ is $a - t$.

Hence, for each $a \in \mathbb{Z}/(k + 1)$, there exists a non- $\mathbf{c}(\tilde{\nu})$ -blocked $\mathbf{c}(\tilde{\lambda})$ -removable a -corner if and only if there exists a non- $\mathbf{c}(\nu)$ -blocked $\mathbf{c}(\lambda)$ -removable $(a - t)$ -corner. Therefore we have $r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)} = r_{\mathbf{c}(\tilde{\nu})\mathbf{c}(\tilde{\lambda})}$. \square

As a corollary of the proof of the above lemma, we have

Corollary 17. *For any $\lambda, \nu \in \mathcal{P}_k$ and $1 \leq t \leq k$ we have $r_{\mathbf{c}(R_t \cup \nu)\mathbf{c}(R_t \cup \lambda)} = r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)}$ or $r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)} + 1$.*

Proof. Take r such that $\lambda_r \geq t > \lambda_{r+1}$ and do a same argument as the above lemma.

Then we have that, if $i \neq r + (k + 1 - t)$, there exists a $\mathbf{c}(\tilde{\nu})$ -nonblocked $\mathbf{c}(\tilde{\lambda})$ -removable a -corner in i -th row if and only if there exists a $\mathbf{c}(\nu)$ -nonblocked $\mathbf{c}(\lambda)$ -removable $(a - t)$ -corner in i' -th row. (Here we put $i' = i$ if $i < r + (k + 1 - t)$ and $i' = i - (k + 1 - t)$ if $i > r + (k + 1 - t)$)

Hence we have $r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)} \leq r_{\mathbf{c}(\tilde{\nu})\mathbf{c}(\tilde{\lambda})} \leq r_{\mathbf{c}(\nu)\mathbf{c}(\lambda)} + 1$. \square

Proposition 18. *For $\lambda \in \mathcal{P}_k$ and $1 \leq t \leq k$, we have $g_{\lambda \cup R_t \cup R_t}^{(k)} = g_{\lambda \cup R_t}^{(k)} \cdot \frac{g_{R_t \cup R_t}^{(k)}}{g_{R_t}^{(k)}}$.*

Proof. Write $\tilde{\mu} = \mu \cup R_t$ for $\mu \in \mathcal{P}_k$.

Define a linear map $\Theta : \Lambda^{(k)} \longrightarrow \Lambda^{(k)}$ by $g_\mu^{(k)} \longmapsto g_\mu^{(k)}$ for all $\mu \in \mathcal{P}_k$ and put $X = \text{span}\{g_\lambda^{(k)} \mid \lambda \in \mathcal{P}_k\}$. Then X is an ideal of $\Lambda^{(k)}$ because $h_r \cdot g_\lambda^{(k)}$ can be written as a linear combination of $\{g_\nu^{(k)} \mid \nu \in \mathcal{P}_k\}$, by (7) and Proposition 12.

Next we claim

$$\Theta|_X \circ (h_r \cdot) = (h_r \cdot) \circ \Theta|_X : X \longrightarrow X$$

for $1 \leq r \leq k$, where $h_r \cdot$ denotes the multiplication by h_r .

Proof of claim.

It suffices to show $h_r \cdot g_{\mu \cup R_t}^{(k)} = \Theta(h_r \cdot g_{\mu}^{(k)})$ for $\mu \in \mathcal{P}_k$. More generally, we can show $h_r \cdot g_{\mu \cup (t)}^{(k)} = \Theta(h_r \cdot g_{\mu \cup (t)}^{(k)})$ for $\mu \in \mathcal{P}_k$:

$$\begin{aligned} h_r \cdot g_{\mu \cup (t)}^{(k)} &= \sum_{s=0}^r (-1)^{r-s} \sum_{\substack{\eta \\ \mathfrak{c}(\eta)/\mathfrak{c}(\mu \cup (t)) \text{ is a weak } s\text{-strip}}} \binom{r}{r-s}^{\mathfrak{c}(\eta), \mathfrak{c}(\mu \cup (t))} g_\eta^{(k)} \\ &= \sum_{s=0}^r (-1)^{r-s} \sum_{\substack{\nu \\ \mathfrak{c}(\nu)/\mathfrak{c}(\mu \cup (t)) \text{ is a weak } s\text{-strip}}} \binom{r}{r-s}^{\mathfrak{c}(\nu), \mathfrak{c}(\mu \cup (t))} g_\nu^{(k)} \\ &= \sum_{s=0}^r (-1)^{r-s} \sum_{\substack{\nu \\ \mathfrak{c}(\nu)/\mathfrak{c}(\mu \cup (t)) \text{ is a weak } s\text{-strip}}} \binom{r}{r-s}^{\mathfrak{c}(\nu), \mathfrak{c}(\mu \cup (t))} g_\nu^{(k)} \\ &= \Theta \left(\sum_{s=0}^r (-1)^{r-s} \sum_{\substack{\nu \\ \mathfrak{c}(\nu)/\mathfrak{c}(\mu \cup (t)) \text{ is a weak } s\text{-strip}}} \binom{r}{r-s}^{\mathfrak{c}(\nu), \mathfrak{c}(\mu \cup (t))} g_\nu^{(k)} \right) \\ &= \Theta \left(h_r \cdot g_{\mu \cup (t)}^{(k)} \right). \end{aligned}$$

Here the second equality uses Proposition 12, and the third equality uses Lemma 16. Hence the claim is proved.

Since h_1, \dots, h_k generate $\Lambda^{(k)}$, the claim implies that $\Theta|_X$ is a $\Lambda^{(k)}$ -module homomorphism. Hence for any $x \in X$,

$$x \cdot \Theta(g_{R_t}^{(k)}) = \Theta(x g_{R_t}^{(k)}) = \Theta(x) \cdot g_{R_t}^{(k)},$$

which implies $\Theta(x) = x \cdot \frac{g_{R_t \cup R_t}^{(k)}}{g_{R_t}^{(k)}}$ for any $x \in X$. Setting $x = g_{R_t \cup \lambda}^{(k)}$ gives the proposition. \square

Theorem 19. Let $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ be as in (NP), and put $Q = R_{t_1} \cup \dots \cup R_{t_m}$. Then, for $\lambda \in \mathcal{P}_k$ we have

$$\frac{g_{P \cup \lambda}^{(k)}}{g_P^{(k)}} = \frac{g_{Q \cup \lambda}^{(k)}}{g_Q^{(k)}}.$$

Proof. Induction on $\sum_i (a_i - 1)$. If $\sum_i (a_i - 1) = 0$ then it is obvious since $P = Q$.

Otherwise, we can assume $a_1 > 1$ without loss of generality. Write $P = R_{t_1} \cup R_{t_1} \cup P'$. By Proposition 18 we have

$$\frac{g_{P' \cup \lambda \cup R_{t_1} \cup R_{t_1}}^{(k)}}{g_{P' \cup \lambda \cup R_{t_1}}^{(k)}} = \frac{g_{R_{t_1} \cup R_{t_1}}^{(k)}}{g_{R_{t_1}}^{(k)}} = \frac{g_{P' \cup R_{t_1} \cup R_{t_1}}^{(k)}}{g_{P' \cup R_{t_1}}^{(k)}},$$

thus we conclude

$$\frac{g_{P' \cup \lambda \cup R_{t_1} \cup R_{t_1}}^{(k)}}{g_{P' \cup R_{t_1} \cup R_{t_1}}^{(k)}} = \frac{g_{P' \cup \lambda \cup R_{t_1}}^{(k)}}{g_{P' \cup R_{t_1}}^{(k)}} = \frac{g_{Q \cup \lambda}^{(k)}}{g_Q^{(k)}}.$$

Here we used induction hypothesis for the second equality. \square

4. A FACTORIZATION OF $g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m} \cup (r)}^{(k)}$

In this section we will give an explicit formula for $g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m} \cup \lambda}^{(k)} / g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}}^{(k)}$ when $\lambda = (r)$.

Roughly speaking, K - k -Schur functions can be calculated by “solving” the system of Pieri rule formulas (7). To solve such a system, it is important to understand concretely what weak strips $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$ are.

If μ is a union of k -rectangles $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}$ the situation is simple: if $\mathfrak{c}(\nu)/\mathfrak{c}(P)$ is a weak strip then ν has the form $P \cup (s)$ for some s , as we will see in the proof of the following proposition. Thus the Pieri rule also has a simple explicit expression as follows:

Proposition 20. *Let P and $\alpha_P(u)$ ($u \in \mathbb{Z}_{>0}$) be as in (NP) in Section 3, before Proposition 14. Then, for $1 \leq r \leq k$, we have*

$$g_P^{(k)} \cdot h_r = \sum_{s=0}^r (-1)^{r-s} \binom{\alpha_P(s+1)}{r-s} g_{P \cup (s)}^{(k)}.$$

Proof. We have $\mathfrak{c}(P) = \underbrace{R_{t_m} \oplus \dots \oplus R_{t_m}}_{a_m} \oplus \dots \oplus \underbrace{R_{t_1} \oplus \dots \oplus R_{t_1}}_{a_1}$ and all addable

corners of $\mathfrak{c}(P)$ has the same residue, say i . Moreover, $\mathfrak{c}(P)$ has a total of $\sum_j a_j$ removable corners, a_j of which are derived from the removable corner of R_{t_j} and having the residue $i + t_j$ for each j .

Next we claim that if $\gamma/\mathfrak{c}(P)$ is a weak s -strip then $\gamma = s_{i+s-1} \dots s_{i+1} s_i(\mathfrak{c}(P))$.

Proof of the claim. We prove it by induction on s . If $s = 1$, it is obvious because all addable corners of $\mathfrak{c}(P)$ have the same residue i .

Let $s > 1$ and $\gamma/\mathfrak{c}(P)$ be a weak s -strip. Then we can write $\gamma = s_{j_s} \dots s_{j_2} s_{j_1}(\mathfrak{c}(P))$, where (j_s, \dots, j_1) is cyclically decreasing (see Definition-Proposition 3(4)).

Since $s_{j_{s-1}} \dots s_{j_2} s_{j_1}(\mathfrak{c}(P))/\mathfrak{c}(P)$ is a weak $(s-1)$ -strip, we have $(j_{s-1}, \dots, j_1) = (i + s - 2, \dots, i + 1, i)$ by the induction hypothesis. Since $(j_s, i + s - 2, \dots, i + 1, i)$ is cyclically decreasing, we have $j_s \notin \{i - 1, i, i + 1, \dots, i + s - 2\}$.

If $j_s \neq i + s - 1$, then s_{j_s} commutes with $s_i, s_{i+1}, \dots, s_{i+s-2}$ and

$$\begin{aligned} \gamma &= s_{j_s} s_{i+s-2} \dots s_{i+1} s_i(\mathfrak{c}(P)) \\ &= s_{i+s-2} \dots s_{i+1} s_i s_{j_s}(\mathfrak{c}(P)). \end{aligned}$$

However, $|s_{j_s}(\mathfrak{c}(P))|_k \leq |\mathfrak{c}(P)|_k$ because $\mathfrak{c}(P)$ doesn't have an addable corner of residue j_s . Hence $|\gamma|_k \leq |\mathfrak{c}(P)|_k + s - 1$, violating the assumption that $\gamma/\mathfrak{c}(P)$ is a weak s -strip.

Hence we have $j_s = i + s - 1$, completing the proof of the claim.

Since $s_{i+s-1} \cdots s_{i+1} s_i(\mathfrak{c}(P))$ has the form below on the right, we can see that the corresponding k -bounded partition has the form $P \cup (s)$.

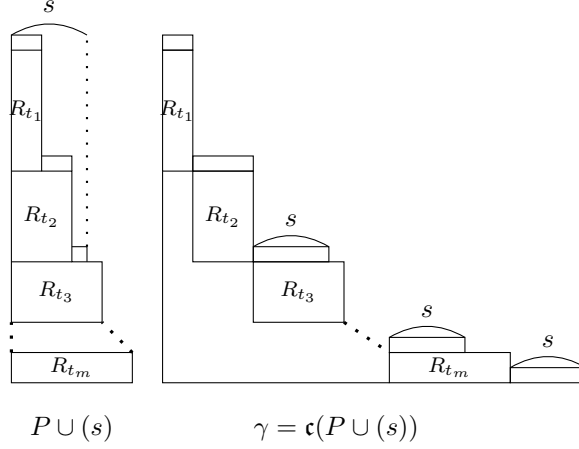


FIGURE 1. the shapes of $P \cup (s)$ and $\mathfrak{c}(P \cup (s))$. In this figure $a_i = 1$ for all i .

Now we get back to the proof of the proposition. Let $\gamma = s_{i+s-1} \cdots s_{i+1} s_i(\mathfrak{c}(P))$. Then the removable corner of $\mathfrak{c}(P)$ corresponding to the removable corner of R_{t_a} is γ -blocked if and only if $s \geq t_a$. Then the number of residues of γ -nonblocked removable corners of $\mathfrak{c}(P)$ is exactly $\alpha_P(s+1)$. \square

The above proposition gives an expression for $g_P^{(k)} h_r$ as a linear combination of $\{g_{P \cup (s)}^{(k)}\}_s$. To solve this linear equation, we need the following lemma of binomial coefficients.

Lemma 21. *Let l be a positive integer and $\beta_1, \beta_2, \dots, \beta_{l+1}$ be integers such that $\beta_i \geq \beta_{i+1} \geq \beta_i - 1$ for each i .*

Let $C = \left((-1)^{r-s} \binom{\beta_{s+1}}{r-s} \right)_{r,s=0}^l$. Then $C^{-1} = \left(\binom{\beta_r + r - s - 1}{r-s} \right)_{r,s=0}^l$.

Here $\binom{a}{b}$ is considered to be 0 if $b < 0$.

Proof. Put $D = \left(\binom{\beta_r + r - s - 1}{r-s} \right)_{r,s=0}^l$. The (p, q) element of the matrix DC is

$$\begin{aligned} (DC)_{pq} &= \sum_{i=0}^l \binom{\beta_p + p - i - 1}{p-i} \cdot (-1)^{i-q} \binom{\beta_{q+1}}{i-q} \\ &= \sum_{i=q}^p \binom{\beta_p + p - i - 1}{p-i} \cdot (-1)^{i-q} \binom{\beta_{q+1}}{i-q} \end{aligned}$$

$$= \sum_{j=0}^{p-q} \binom{\beta_p + p - q - j - 1}{p - q - j} \cdot (-1)^j \binom{\beta_{q+1}}{j},$$

which is 0 unless $p \geq q$.

Let us consider the case $p \geq q$.

By applying the next lemma for $a = \beta_{q+1}$, $b = \beta_p$, $c = p - q \geq 0$, we have

$$\begin{aligned} (DC)_{pq} &= (-1)^{p-q} \binom{\beta_{q+1} - \beta_p}{p - q} \\ &= \begin{cases} 0 & (\text{if } p > q), \\ 1 & (\text{if } p = q), \end{cases} \end{aligned}$$

where the last equality follows from $\beta_{q+1} - \beta_p \in \{0, 1, \dots, p - q - 1\}$ (if $q + 1 \leq p$). \square

Lemma 22. *For integers a, b and a nonnegative integer c ,*

$$\sum_{i=0}^c (-1)^i \binom{a}{i} \binom{b - 1 + c - i}{c - i} = (-1)^c \binom{a - b}{c}.$$

Proof. Since $\binom{m}{n}$ is the coefficient of X^n in $(1 + X)^m \in \mathbb{Z}[[X]]$ for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} (\text{LHS}) &= \sum_{i=0}^c (-1)^i \binom{a}{i} \binom{-b}{c - i} \\ &= (-1)^c (\text{the coefficient of } X^c \text{ in } (1 + X)^a (1 + X)^{-b} \in \mathbb{Z}[[X]]) \\ &= (-1)^c \binom{a - b}{c}. \end{aligned}$$

\square

Now we can deduce the formula showing the goal of this section.

Theorem 23. *If $P, \alpha_P(u)$ and r are as in Proposition 20, then we have*

$$\frac{g_{P \cup (r)}^{(k)}}{g_P^{(k)}} = \sum_{s=0}^r \binom{\alpha_P(r) + r - s - 1}{r - s} h_s.$$

In particular, if $t_m < r$, which means $\alpha_P(r) = 0$, we have

$$\frac{g_{P \cup (r)}^{(k)}}{g_P^{(k)}} = h_r = g_{(r)}^{(k)}$$

On the other hand, when $m = 1$,

$$\frac{g_{R_t \cup (r)}^{(k)}}{g_{R_t}^{(k)}} = \begin{cases} h_r & (\text{if } r > t), \\ h_r + h_{r-1} + \dots + h_0 & (\text{if } r \leq t). \end{cases}$$

Proof. Apply Lemma 21 for Proposition 20. \square

5. A FACTORIZATION OF $g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m} \cup \lambda}^{(k)}$ WITH SMALL λ AND SPLITTING

$$g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}}^{(k)} \text{ INTO } g_{R_{t_1}^{a_1}}^{(k)} \cdots g_{R_{t_m}^{a_m}}^{(k)}$$

5.1. **Statements.** Our goal in this section is to show the equality

$$g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}}^{(k)} = g_{R_{t_1}^{a_1}}^{(k)} \cdots g_{R_{t_m}^{a_m}}^{(k)}$$

for $1 \leq t_1 < \dots < t_m \leq k$ and $a_i > 0$ (see Theorem 31).

The essential part is to prove $g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}}^{(k)} = g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_{m-1}}^{a_{m-1}}}^{(k)} g_{R_{t_m}^{a_m}}^{(k)}$, and the remaining part follows from the results from Section 3 and induction (on n). This is a simple statement, but our proof involves an induction on the shape of partitions, thus we have to prove a more general statement (see the case $t_n < r$ of part (2) of Theorem 30): Let $P = \bigcup_{i=1}^m R_{t_i}^{a_i}$ be as in (NP), Section 3, before Proposition 14, and λ as follows:

(N λ) Let $(\emptyset \neq) \lambda \in \mathcal{P}_k$ with satisfying $\bar{\lambda} \subset R'_l$, where we write $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)-1})$ and $\bar{l} = l(\bar{\lambda}) = l(\lambda) - 1$. (Here we consider R_t to be \emptyset unless $1 \leq t \leq k$)

(Note: when $l(\lambda) = 1$, we have $\bar{l} = 0$ and $\bar{\lambda} = \emptyset = R'_l$ thus λ satisfies (N λ). When $l(\lambda) > k + 1$, we have $\bar{l} > k$ and $\bar{\lambda} \neq \emptyset = R'_l$ thus λ does not satisfy (N λ).)

Then,

$$(9) \quad g_{P \cup \lambda}^{(k)} = g_P^{(k)} g_{\lambda}^{(k)} \quad \text{when } \lambda_{l(\lambda)} > \max_i \{t_i\}.$$

5.2. **Proofs.** We will prove a slightly even more general formula than (9) (see part (2) of Theorem 30) in the following procedure.

• Step (A):

First we write $g_{\lambda}^{(k)}$ as a linear combination of products of h_i 's and $g_{\mu}^{(k)}$'s with $l(\mu) < l(\lambda)$: putting $\bar{\lambda} = (\lambda_1, \dots, \lambda_{l(\lambda)-1})$, we have

$$g_{\lambda}^{(k)} = \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R_{k-l(\bar{\lambda})+1} \\ \mu/\bar{\lambda} : \text{vertical strip}}} (-1)^{|\mu/\bar{\lambda}|} g_{\mu}^{(k)} \sum_{i \geq 0} \binom{(|\mu/\bar{\lambda}| + r_{\mu'\bar{\lambda}'} + i - 1)}{i} h_{\lambda_{l(\lambda)} - |\mu/\bar{\lambda}| - i}$$

if $\bar{\lambda}_1 + l(\bar{\lambda}) \leq k + 1$ (Lemmas 26 (1), 28 (1), 29 (1) and 34).

• Step (B):

Derive a similar expression for $g_{P \cup \lambda}^{(k)}$ (parts Lemmas 26 (2)-(3), 28 (2)-(3), 29 (2)-(3), 34).

• Step (C):

Compare (B) with the equality obtained by multiplying the formula in Step (A) by $g_P^{(k)}$, noticing $g_P^{(k)} g_{\mu}^{(k)} = g_{P \cup \mu}^{(k)}$ by induction.

Step (A) consists of two substeps:

- Step (A-1): Write down the Pieri rule for $g_{\mu}^{(k)} h_r$ explicitly.
- Step (A-2): Solve the system of Pieri rule formulas to give expressions for $g_{\lambda}^{(k)}$ as a linear combination of $\{g_{\mu}^{(k)} h_r\}_{\mu, r}$.

Obtaining an expression for $g_{P \cup \lambda}^{(k)}$ in Step (B) follows from similar steps (B-1) and (B-2).

- Step (B-1): Write down the Pieri rule for $g_{P \cup \mu}^{(k)} h_r$ explicitly.

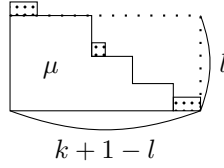
- Step (B-2): Solve the system of Pieri rule formulas to give expressions for $g_{P \cup \lambda}^{(k)}$ as a linear combination of $\{g_{P \cup \mu}^{(k)} h_r\}_{\mu, r}$.

5.3. Steps (A-1) and (B-1). Toward Step (A-1) and (B-1), let us begin with describing weak strips $\mathbf{c}(\lambda)/\mathbf{c}(\mu)$ where μ is contained in a k -rectangle.

Lemma 24. Assume $\mu \subset R_{k+1-l}$ and $\mu_l > 0$. Let $0 \leq u \leq \mu_l$ be an integer.

(1) For $\kappa \in \mathcal{P}_k$,

$$\begin{aligned} \mathbf{c}(\kappa)/\mathbf{c}(\mu) \text{ is a weak } u\text{-strip} &\iff \begin{cases} \kappa/\mu \text{ is a horizontal } u\text{-strip,} \\ \kappa_1 \leq k - l + 1, \end{cases} \\ &\iff \kappa = \nu \cup (s), \\ &\text{where } \begin{cases} \nu \subset R_{k+1-l}, \\ \nu/\mu \text{ is a horizontal strip of size } \leq u, \\ s = u - |\nu/\mu|. \end{cases} \end{aligned}$$



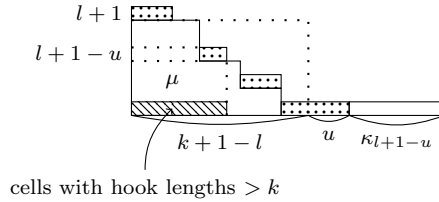
(2) For $\tilde{\kappa} \in \mathcal{P}_k$,

$$\begin{aligned} \mathbf{c}(\tilde{\kappa})/\mathbf{c}(P \cup \mu) \text{ is a weak } u\text{-strip} \\ \iff \tilde{\kappa} = P \cup \kappa, \text{ where } \mathbf{c}(\kappa)/\mathbf{c}(\mu) \text{ is a weak } u\text{-strip} \\ \iff \tilde{\kappa} = P \cup \nu \cup (x), \text{ where } \begin{cases} \nu \subset R_{k+1-l}, \\ \nu/\mu: \text{ horizontal strip,} \\ |\nu/\mu| + x = u. \end{cases} \end{aligned}$$

Proof. (1): The second equivalence is obvious.

The “if” part of the first equivalence is easy: since $\kappa_1 \leq k+1-l$ and $l(\kappa) \leq l+1$, we have $\mathbf{c}(\kappa) = \kappa$ or $\mathbf{c}(\kappa) = (\kappa_1 + \kappa_{l+1}, \kappa_2, \dots)$ and hence $\kappa^{\omega_k}/\mu^{\omega_k}$ is a vertical strip.

Hence it suffices to prove the “only if” part of the first equivalence: let $\mu \subset R_{k+1-l}$, $\mu_l > 0$, and $\mathbf{c}(\kappa)/\mathbf{c}(\mu)$ be a weak strip of size $\leq \mu_l$, and we shall prove $\kappa_1 \leq k+1-l$.



Assume, on the contrary, that $\kappa_1 > k+1-l$. Write $\bar{\kappa} = (\kappa_2, \kappa_3, \dots) \subset R_{k+1-l}$. Then by Lemma 1 we have

$$\mathbf{c}(\kappa)_i = \mathbf{c}(\bar{\kappa})_{i-1} = \bar{\kappa}_{i-1} = \kappa_i \quad (\text{for } i > 1), \text{ and}$$

$$\mathbf{c}(\kappa)_1 = \kappa_1 + \underbrace{\mathbf{c}(\kappa)_{1+k+1-\kappa_1}}_{\substack{< l+1 \\ \geq \kappa_l \geq \mu_l}} \geq \kappa_1 + \mu_l > k+1-l + \mu_l.$$

Hence, the hook lengths of $(1, 1), \dots, (1, \mu_l)$ in $\mathbf{c}(\kappa)$ are all greater than k because

$$\begin{aligned} h_{(1,j)}(\mathbf{c}(\kappa)) &= \mathbf{c}(\kappa)_1 + \mathbf{c}(\kappa)'_j - 1 - j + 1 \\ &> k+1-l + \mu_l + \underbrace{\mathbf{c}(\kappa)'_j}_{\geq \kappa'_j \geq \mu'_j \geq l \geq -\mu_l} - j \\ &\geq k+1 \end{aligned}$$

for $1 \leq j \leq \mu_l$. On the other hand, those of $(2, 1), \dots, (2, \mu_l)$ in $\mathbf{c}(\kappa)$ are less than or equal to k because $\bar{\kappa} \subset \mu \subset R_{k+1-l}$.

Hence

$$\kappa_j^{\omega_k} = \mathbf{c}(\kappa)'_j - 1 = \kappa'_j - 1$$

for $1 \leq j \leq \mu_l$.

Since $\mathbf{c}(\kappa)/\mathbf{c}(\mu)$ is a weak strip, $\kappa^{\omega_k}/\mu^{\omega_k}$ is a vertical strip. Hence

$$\kappa'_j - 1 = \kappa_j^{\omega_k} \geq \mu_j^{\omega_k} = \mu'_j = l$$

for $1 \leq j \leq \mu_l$, which implies $\kappa_{l+1} \geq \mu_l$. Then we have

$$|\kappa/\mu| \geq (\underbrace{\kappa_{l+1} - \mu_{l+1}}_{=0}) + (\underbrace{\kappa_1 - \mu_1}_{>0}) > \mu_l$$

since $\kappa_1 > k+1-l \geq \mu_1$. This is a contradiction.

(2): The first equivalence follows from Corollary 12. The second equivalence follows from (1). \square

Next let us explicitly describe the weak Pieri rule (7), after we prepare a notation for convenience.

Definition 25. Let P and $\alpha_P(u)$ ($u \in \mathbb{Z}_{>0}$) be as in (NP). For $\nu \subset R_{k+1-l}(\nu)$, $0 \leq u \leq \nu_l(\nu)$, $p \in \mathbb{Z}$, we set

$$\begin{aligned} T_{\nu, u, p} &:= \sum_{s=0}^u (-1)^s \binom{p}{s} g_{\nu \cup (u-s)}^{(k)}, \\ T'_{P, \nu, u, p} &:= \sum_{s=0}^u (-1)^s \binom{p + \alpha_P(u+1-s)}{s} g_{P \cup \nu \cup (u-s)}^{(k)}. \end{aligned}$$

Lemma 26. Let P and $\alpha_P(u)$ ($u \in \mathbb{Z}_{>0}$) be as in (NP). Assume $\mu \subset R_{k+1-l}$, $\mu_l > 0$, $\mu_l \geq r \geq 0$. Then we have

(1)

$$\begin{aligned} g_\mu^{(k)} h_r &= \sum_{\substack{\nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} \sum_{s=0}^{r-|\nu/\mu|} (-1)^s \binom{r_{\nu\mu}}{s} g_{\nu \cup (r-|\nu/\mu|-s)}^{(k)} \\ &\left(= \sum_{\substack{\nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} T_{\nu, r-|\nu/\mu|, r_{\nu, \mu}} \right). \end{aligned}$$

(2) If $\max_i(t_i) < \mu_l$,

$$g_{P \cup \mu}^{(k)} h_r = \sum_{\substack{\nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} \sum_{s=0}^{r-|\nu/\mu|} (-1)^s \binom{r_{\nu\mu} + \alpha_P(r - |\nu/\mu| + 1 - s)}{s} g_{P \cup \nu \cup (r - |\nu/\mu| - s)}^{(k)} \\ \left(= \sum_{\substack{\nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} T'_{P, \nu, r - |\nu/\mu|, r_{\nu, \mu}} \right).$$

(3) If $\max_i(t_i) = \mu_l$,

$$g_{P \cup \mu}^{(k)} h_r = \sum_{\substack{\nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} \sum_{s=0}^{r-|\nu/\mu|} (-1)^s \binom{r_{\nu\mu} + \alpha_P(r - |\nu/\mu| + 1 - s) - 1}{s} g_{P \cup \nu \cup (r - |\nu/\mu| - s)}^{(k)} \\ \left(= \sum_{\substack{\nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} T'_{P, \nu, r - |\nu/\mu|, r_{\nu, \mu} - 1} \right).$$

Proof. (1) We transform the right-hand side of Eq. (7), Proposition 9, into the right-hand side of part (1) of the Lemma as follows:

$$(10) \quad g_{\mu}^{(k)} h_r = \sum_{u=0}^r (-1)^{r-u} \sum_{\substack{\kappa \\ \mathbf{c}(\kappa)/\mathbf{c}(\mu): \text{weak } u\text{-strip}}} \binom{r_{\mathbf{c}(\kappa)\mathbf{c}(\mu)}}{r-u} g_{\kappa}^{(k)} \\ = \sum_{u=0}^r (-1)^{r-u} \sum_{\substack{\nu \text{ s.t.} \\ \nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s. of size } \leq u}} \binom{r_{\mathbf{c}(\nu \cup (u - |\nu/\mu|)), \mathbf{c}(\mu)}}{r-u} g_{\nu \cup (u - |\nu/\mu|)}^{(k)} \\ = \sum_{\substack{\nu \text{ s.t.} \\ \nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} \sum_{u=|\nu/\mu|}^r (-1)^{r-u} \binom{r_{\mathbf{c}(\nu \cup (u - |\nu/\mu|)), \mathbf{c}(\mu)}}{r-u} g_{\nu \cup (u - |\nu/\mu|)}^{(k)} \\ = \sum_{\substack{\nu \text{ s.t.} \\ \nu \subset R_{k+1-l} \\ \nu/\mu: \text{h.s.}}} \sum_{s=0}^{r-|\nu/\mu|} (-1)^s \binom{r_{\nu, \mu}}{s} g_{\nu \cup (r - s - |\nu/\mu|)}^{(k)}.$$

Here, the equality (i) uses Lemma 24 (1) in order to change the summation variable from κ to ν according to $\kappa = \nu \cup (u - |\nu/\mu|)$.

For the equality (ii) we use (1) of the following Lemma 27 and put $s = r - u$. Note that $u - |\nu/\mu| \geq \mu_l$ occurs only if $u - |\nu/\mu| = u = r = \mu_l$ since $u \leq r \leq \mu_l$, in which case we have $\binom{r_{\nu \cup (u - |\nu/\mu|), \mu}}{r-u} = 1 = \binom{r_{\nu, \mu}}{r-u}$.

We can prove (2) and (3) almost the same as (1), using Lemma 24 (2) for (i), and (2) of the following Lemma 27 for (ii).

Note that, in the same way as (1), the case $u - |\nu/\mu| \geq \mu_l$ and $\max_i(t_i) < \mu_l$ appears in the expression for $g_{P \cup \lambda}^{(k)}$ corresponding to (10) only in the form $\binom{r_{\nu\mu} + \alpha_P(\mu_l + 1) - 1}{0}$, which is equal to $\binom{r_{\nu\mu} + \alpha_P(\mu_l + 1)}{0}$.

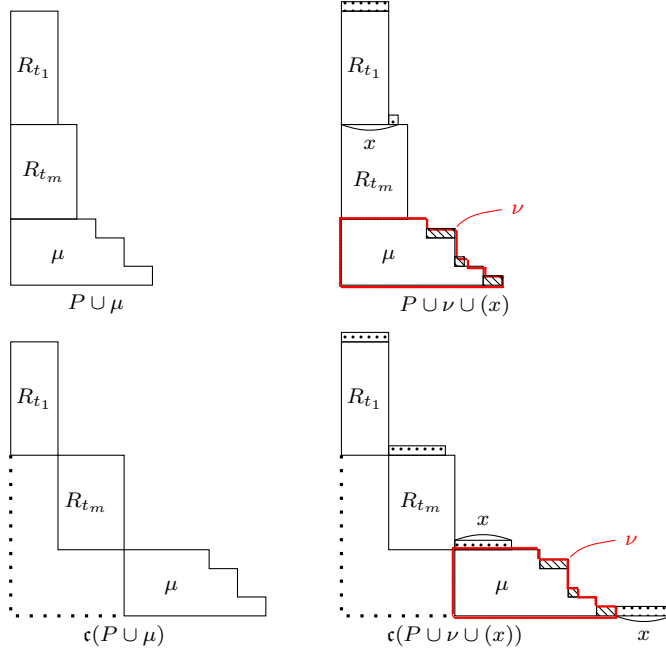
□

Lemma 27. Let μ be as in Lemma 26. Let $\mu \subset \nu \subset R_{k+1-l}$ and assume ν/μ is a horizontal strip. Let $0 \leq x \leq \nu_l$. Then we have

$$(1) \quad r_{\mathfrak{c}(\nu \cup (x)), \mathfrak{c}(\mu)} = r_{\nu, \mu} - \delta[x \geq \mu_l].$$

(2) Let P and $\alpha_P(u)$ ($u \in \mathbb{Z}_{>0}$) be as in Lemma 26 and assume $\max_i(t_i) \leq \mu_l$. Then

$$r_{\mathfrak{c}(P \cup \nu \cup (x)), \mathfrak{c}(P \cup \mu)} = \begin{cases} r_{\nu, \mu} + \alpha_P(x+1) - \delta[x \geq \mu_l] & (\text{if } \max_i(t_i) < \mu_l), \\ r_{\nu, \mu} + \alpha_P(x+1) - 1 & (\text{if } \max_i(t_i) = \mu_l). \end{cases}$$



Proof. (1): Since $\mu \subset R_{k+1-l}$ we have $\mathfrak{c}(\mu) = \mu$. Since $\nu \subset R_{k+1-l}$ and $x \leq \nu_l$, we have $\mathfrak{c}(\nu \cup (x))_i = (\nu \cup (x))_i$ for $i \neq 1$. Thus $r_{\mathfrak{c}(\nu \cup (x)), \mathfrak{c}(\mu)} = r_{\nu \cup (x), \mu}$. Moreover, $r_{\nu \cup (x), \mu} \neq r_{\nu, \mu}$ happens only if the $(l+1)$ -th part of $\nu \cup (x)$ blocks the μ -removable corner in the l -th row, i.e. $x \geq \mu_l$, in which case $r_{\nu \cup (x), \mu} = r_{\nu, \mu} - 1$.

(2):

Assume $t_1 < \dots < t_m$ without loss of generality and thus $\max_i(t_i) = t_m$. We put $T = \mathfrak{c}(P)_1 = \sum_j t_j$. Since $\mathfrak{c}(P \cup \mu) = \mu \oplus \mathfrak{c}(P)$, a removable corner (r, c) of $\mathfrak{c}(P \cup \mu)$ satisfies one of the following:

- (type 1) $r \geq l+1$, and $(r-l, c)$ is a removable corner of $\mathfrak{c}(P)$,
- (type 2) $c \geq T+1$, and $(r, c-T)$ is a removable corner of μ .

We put

$$X_j := \text{Res}\{\text{removable corners of } \mathfrak{c}(P \cup \mu) \text{ of type } j\}$$

$$Y_j := \text{Res}\{\mathfrak{c}(P \cup \nu \cup (x))\text{-nonblocked removable corners of } \mathfrak{c}(P \cup \mu) \text{ of type } j\}$$

for $j = 1, 2$.

We denote by i the residue of top addable corner of $\mathbf{c}(P \cup \mu)$. Then we have $X_1 = \{i + t_1, i + t_2, \dots, i + t_m\}$ and $i + \mu_l \in X_2 \subset [i + \mu_l, i + k - 1]$. Note that

$$\{i + t_1, \dots, i + t_m\} \cap [i + \mu_l, i + k - 1] = \begin{cases} \emptyset & (\text{if } t_m < \mu_l), \\ \{i + \mu_l\} & (\text{if } t_m = \mu_l). \end{cases}$$

Next we show that, for $i \geq 1$,

$$(11) \quad \mathbf{c}(P \cup \nu \cup (x))_{l+i} = \mathbf{c}(P \cup (x))_i,$$

$$(12) \quad \mathbf{c}(P \cup \nu \cup (x))'_{T+i} = \mathbf{c}(\nu \cup (x))'_i.$$

(11) is obvious since the smallest part of ν , which is ν_l , is greater than or equal to the largest part of $P \cup (x)$, which is $\max\{x, t_m\}$.

For (12), first we note that, by (11) and Figure 1 in the proof of Proposition 20, we have

$$\begin{aligned} \mathbf{c}(P \cup \nu \cup (x))_{l+1} &= \mathbf{c}(P \cup (x))_1 = T + x, \\ \mathbf{c}(P \cup \nu \cup (x))_{l+2} &= \mathbf{c}(P \cup (x))_2 = T, \\ &\vdots \\ \mathbf{c}(P \cup \nu \cup (x))_{l+k+1-\mu_l} &= \mathbf{c}(P \cup (x))_{k+1-\mu_l} = T. \end{aligned}$$

Then by Lemma 1 we have, for $1 \leq i \leq l$,

$$\begin{aligned} \mathbf{c}(P \cup \nu \cup (x))_i &= \mathbf{c}(P \cup \nu \cup (x))_{i+(k+1-\nu_i)} + \nu_i \\ &= \begin{cases} \mathbf{c}(P \cup \nu \cup (x))_{l+1} + \nu_1 = T + x + \nu_1 & (\text{if } i = 1 \text{ and } \nu_1 = k + 1 - l), \\ T + \nu_i & (\text{otherwise}), \end{cases} \end{aligned}$$

where we used $(P \cup \nu \cup (x))_i = \nu_i$ for $1 \leq i \leq l$ for the first equality and $l + 1 \leq i + (k + 1 - \nu_i) \leq l + (k + 1 - \mu_n)$ (the first equality holds if and only if $i = 1$ and $\nu_1 = k + 1 - l$) for the second equality.

Thus we have $\mathbf{c}(P \cup \nu \cup (x))_i = \mathbf{c}(\nu \cup (x))_i + T$ for $1 \leq i \leq l + 1$ and $\mathbf{c}(P \cup \nu \cup (x))_i \leq T$ for $i > l + 1$, which implies (12).

Hence, $|Y_1| = r_{\mathbf{c}(P \cup (x)), \mathbf{c}(P)} = \alpha_P(x + 1)$, and $|Y_2| = r_{\mathbf{c}(\nu \cup (x)), \mu} = r_{\nu\mu} - \delta[x \geq \mu_l]$. Moreover $Y_1 \cap Y_2 = \{i + \mu_l\}$ if $x < t_m = \mu_l$, and $Y_1 \cap Y_2 = \emptyset$ otherwise. Then

$$\begin{aligned} r_{\mathbf{c}(P \cup \nu \cup (x)), \mathbf{c}(P \cup \mu)} &= |Y_1| + |Y_2| - |Y_1 \cap Y_2| \\ &= \begin{cases} \alpha_P(x + 1) + r_{\nu\mu} - \delta[x \geq \mu_l] & (\text{if } t_m < \mu_l), \\ \alpha_P(x + 1) + r_{\nu\mu} - \underbrace{\delta[x \geq \mu_l] - \delta[x < t_m]}_{=-1} & (\text{if } t_m = \mu_l). \end{cases} \end{aligned}$$

□

Thus Steps (A-1) and (B-1) have been achieved.

5.4. Steps (A-2) and (B-2). The next lemma is technically important to perform the instructions in Step (A-2) and (B-2).

Lemma 28. *Let ν, u, p be as in the assumptions in Definition 25 and n be an integer. Then we have the following equalities. In particular, in either case, the left-hand side does not depend on p .*

$$\begin{aligned}
(1) \quad & \sum_{i=0}^u \binom{p+n+i-1}{i} T_{\nu, u-i, p} = \sum_{s=0}^u \binom{n+s-1}{s} g_{\nu \cup (u-s)}^{(k)}. \\
(2) \quad & \sum_{i=0}^u \binom{p+n+i-1}{i} T'_{P, \nu, u-i, p} = \sum_{s=0}^u \binom{n - \alpha_P(u+1-s) + s - 1}{s} g_{P \cup \nu \cup (u-s)}^{(k)}.
\end{aligned}$$

Proof. Since both equality can be proved in a parallel manner, we prove (2) here. By the definition of $T'_{P, \nu, u-i, p}$, we have

$$(\text{LHS}) = \sum_{i=0}^u \binom{p+n+i-1}{i} \sum_{s=0}^{u-i} (-1)^s \binom{p + \alpha_P(u-i+1-s)}{s} g_{P \cup \nu \cup (u-i-s)}^{(k)},$$

then putting $t = i + s$,

$$= \sum_{t=0}^u \left(\sum_{s=0}^t \binom{p+n-1+t-s}{t-s} (-1)^s \binom{p + \alpha_P(u+1-t)}{s} \right) g_{P \cup \nu \cup (u-t)}^{(k)},$$

then using Lemma 22,

$$\begin{aligned}
&= \sum_{t=0}^u (-1)^t \binom{-n + \alpha_P(u+1-t)}{t} g_{P \cup \nu \cup (u-t)}^{(k)} \\
&= \sum_{t=0}^u \binom{n - \alpha_P(u+1-t) + t - 1}{t} g_{P \cup \nu \cup (u-t)}^{(k)}.
\end{aligned}$$

□

Now we can express $g_{\lambda}^{(k)}$ (resp. $g_{P \cup \lambda}^{(k)}$) as a linear combination of $g_{\mu}^{(k)} h_r$ (resp. $g_{P \cup \mu}^{(k)} h_r$) as proposed in the description of Step (A-2) (resp. (B-2)).

Lemma 29. *Let P and $\alpha_P(u)$ ($u \in \mathbb{Z}_{>0}$) be as in (NP). Let $\lambda, \bar{\lambda}, \bar{l}$ be as in (N λ) in Section 5.1. Write $r = \lambda_{l(\lambda)}$. Assume that $\bar{l} \geq 1$ and $\max_i(t_i) \leq \bar{l}$.*

(1) *We have*

$$g_{\lambda}^{(k)} = \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_l}} \sum_{q \in \mathbb{Z}} A_{\mu, \bar{\lambda}, q} g_{\mu}^{(k)} \sum_{i \geq 0} \binom{q+i-1}{i} h_{r-|\mu/\bar{\lambda}|-i}.$$

(2) *If $\max_i(t_i) < \bar{l}$, we have*

$$g_{P \cup \lambda}^{(k)} = \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_l}} \sum_{q \in \mathbb{Z}} A_{\mu, \bar{\lambda}, q} g_{P \cup \mu}^{(k)} \sum_{i \geq 0} \binom{q+i+\alpha_P(r)-1}{i} h_{r-|\mu/\bar{\lambda}|-i}.$$

(3) *If $\max_i(t_i) = \bar{l}$, we have*

$$g_{P \cup \lambda}^{(k)} = \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_l}} \sum_{q \in \mathbb{Z}} A_{\mu, \bar{\lambda}, q} g_{P \cup \mu}^{(k)} \sum_{i \geq 0} \binom{q+i+\alpha_P(r)-2+\delta[\mu_{\bar{l}} \neq \bar{l}]}{i} h_{r-|\mu/\bar{\lambda}|-i}.$$

Here, in all of the three expressions, the number $A_{\mu, \bar{\lambda}, q}$ is defined by the following recursion formula:

$$A_{\bar{\lambda}, \bar{\lambda}, q} = \delta_{q, r_{\bar{\lambda}}},$$

$$A_{\mu, \bar{\lambda}, q} = - \sum_{\substack{\mu/\kappa: \text{ h.s.} \\ \bar{\lambda} \subset \kappa \subsetneq \mu}} A_{\kappa, \bar{\lambda}, q - (r_{\mu\mu} - r_{\mu\kappa})} \quad \text{for } \bar{\lambda} \subsetneq \mu \subset R'_i.$$

Notice that for each μ , $A_{\mu, \bar{\lambda}, q} = 0$ except for finitely many q . The explicit value of $A_{\mu, \bar{\lambda}, q}$ will be given in Lemma 34 below.

Remark. In the above recursion formula, $r_{\mu\mu} - r_{\mu\kappa} \geq 0$ always holds because there must be a μ -removable corner in every row in which there is a μ -nonblocked κ -removable corner since μ/κ is a horizontal strip.

Proof. (1) By Lemma 26(1),

$$(\text{RHS}) = \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_i}} \sum_q A_{\mu, \bar{\lambda}, q} \sum_{\substack{\mu \subset \nu \subset R'_i \\ \nu/\mu: \text{ h.s.}}} \sum_{i \geq 0} \binom{q+i-1}{i} T_{\nu, r - |\mu/\bar{\lambda}| - i - |\nu/\mu|, r_{\nu\mu}},$$

then splitting the third summation according to whether $\mu = \nu$ or $\mu \subsetneq \nu$,

$$\begin{aligned} &= \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_i}} \sum_q A_{\mu, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q+i-1}{i} T_{\mu, r - |\mu/\bar{\lambda}| - i, r_{\mu\mu}} \\ &+ \sum_{\substack{\mu, \nu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subsetneq \nu \subset R'_i \\ \nu/\mu: \text{ h.s.}}} \sum_q A_{\mu, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q+i-1}{i} T_{\nu, r - |\nu/\bar{\lambda}| - i, r_{\nu\mu}}, \end{aligned}$$

then replacing the variable μ for the first summation by ν , and splitting it again according to whether $\bar{\lambda} = \nu$ or $\bar{\lambda} \subsetneq \nu$, and rearranging the summands,

$$\begin{aligned} &= \sum_q A_{\bar{\lambda}, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q+i-1}{i} T_{\bar{\lambda}, r - i, r_{\bar{\lambda}\bar{\lambda}}} \\ &+ \sum_{\substack{\nu \text{ s.t.} \\ \bar{\lambda} \subsetneq \nu \subset R'_i}} \underbrace{\left(\sum_q A_{\nu, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q+i-1}{i} T_{\nu, r - |\nu/\bar{\lambda}| - i, r_{\nu\nu}} \right)}_{(X)} \\ &\quad + \underbrace{\sum_q \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subsetneq \nu \\ \nu/\mu: \text{ h.s.}}} A_{\mu, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q+i-1}{i} T_{\nu, r - |\nu/\bar{\lambda}| - i, r_{\nu\mu}}}_{(Y)}. \end{aligned}$$

Then, by the definition of $A_{\nu, \bar{\lambda}, q}$, noting that $\bar{\lambda} \subsetneq \nu$,

$$(X) = - \sum_q \sum_{\substack{\mu \text{ s.t.} \\ \nu/\mu: \text{ h.s.} \\ \bar{\lambda} \subset \mu \subsetneq \nu}} A_{\mu, \bar{\lambda}, q - (r_{\nu\nu} - r_{\nu\mu})} \sum_{i \geq 0} \binom{q+i-1}{i} T_{\nu, r - |\nu/\bar{\lambda}| - i, r_{\nu\nu}},$$

then replacing q by $q + (r_{\nu\nu} - r_{\nu\mu})$,

$$= - \sum_q \sum_{\substack{\mu \text{ s.t.} \\ \nu/\mu: \text{ h.s.} \\ \bar{\lambda} \subset \mu \subsetneq \nu}} A_{\mu, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q + r_{\nu\nu} - r_{\nu\mu} + i - 1}{i} T_{\nu, r - |\nu/\bar{\lambda}| - i, r_{\nu\nu}},$$

then using the independence of the LHS on p of Lemma 28(1) (note that the range of i can be limited to $0 \leq i \leq r - |\nu/\bar{\lambda}|$ since i originally occurs in $h_{r - |\mu/\bar{\lambda}| - i}$ in the statement of part (1) of the Lemma),

$$\begin{aligned} &= - \sum_q \sum_{\substack{\mu \text{ s.t.} \\ \nu/\mu: \text{ h.s.} \\ \bar{\lambda} \subset \mu \subsetneq \nu}} A_{\mu, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q + i - 1}{i} T_{\nu, r - |\nu/\bar{\lambda}| - i, r_{\nu\mu}} \\ &= -(Y). \end{aligned}$$

Hence,

$$\begin{aligned} (\text{RHS}) &= \sum_q A_{\bar{\lambda}, \bar{\lambda}, q} \sum_{i \geq 0} \binom{q + i - 1}{i} T_{\bar{\lambda}, r - i, r_{\bar{\lambda}\bar{\lambda}}} \\ &= \sum_{i \geq 0} \binom{r_{\bar{\lambda}\bar{\lambda}} + i - 1}{i} T_{\bar{\lambda}, r - i, r_{\bar{\lambda}\bar{\lambda}}}, \end{aligned}$$

again by Lemma 28(1), noting that $\binom{0+s-1}{s}$ vanishes unless $s = 0$,

$$= g_{\bar{\lambda}}^{(k)}.$$

(3) is proved almost parallel to (1): By Lemma 26(2) and (3),

$$\begin{aligned} (\text{RHS}) &= \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_t}} \sum_q A_{\mu, \bar{\lambda}, q} \sum_{\substack{\mu \subset \nu \subset R'_t \\ \nu/\mu: \text{ h.s.}}} \sum_{i \geq 0} \binom{q + i + \alpha_P(r) - 2 + \delta[\mu_l \neq \bar{\lambda}_l]}{i} T'_{P, \nu, r - |\mu/\bar{\lambda}| - i - |\nu/\mu|, r_{\nu\mu} - \delta[\mu_l = \bar{\lambda}_l]}, \end{aligned}$$

then, by Lemma 28(2), shifting p by $\delta[\mu_l = \bar{\lambda}_l]$ and noting that $\delta[\mu_l \neq \bar{\lambda}_l] + \delta[\mu_l = \bar{\lambda}_l] = 1$,

$$= \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_t}} \sum_q A_{\mu, \bar{\lambda}, q} \sum_{\substack{\mu \subset \nu \subset R'_t \\ \nu/\mu: \text{ h.s.}}} \sum_{i \geq 0} \binom{q + i + \alpha_P(r) - 1}{i} T'_{P, \nu, r - |\nu/\bar{\lambda}| - i, r_{\nu\mu}}.$$

Note that the following deformation is also valid for the case $\max_i(t_i) < \bar{\lambda}_l$. Applying the same argument as (1),

$$= \sum_{i \geq 0} \binom{r_{\bar{\lambda}\bar{\lambda}} + \alpha_P(r) - 1 + i}{i} T'_{P, \bar{\lambda}, r - i, r_{\bar{\lambda}\bar{\lambda}}},$$

then by Lemma 28(2),

$$\begin{aligned} &= \sum_{s \geq 0} \binom{-\alpha_P(r+1-s) + \alpha_P(r) - 1 + s}{s} g_{P \cup \bar{\lambda} \cup (r-s)}^{(k)} \\ &= g_{P \cup \lambda}^{(k)}. \end{aligned}$$

Here the last equality follows from $\binom{-\alpha_P(r+1-s) + \alpha_P(r) - 1 + s}{s} = (-1)^s \binom{\alpha_P(r+1-s) - \alpha_P(r)}{s}$ and $0 \leq \alpha_P(r+1-s) - \alpha_P(r) \leq s-1$ for $s \geq 1$.

For (2), we have

$$(\text{RHS}) = \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_l}} \sum_q A_{\mu, \bar{\lambda}, q} \sum_{\substack{\mu \subset \nu \subset R'_l \\ \nu/\mu: \text{ h.s.}}} \sum_{i \geq 0} \binom{q+i+\alpha_P(r)-1}{i} T'_{P, \nu, r-|\mu/\bar{\lambda}|-i-|\nu/\mu|, r_{\nu\mu}},$$

which is equal to $g_{P \cup \lambda}^{(k)}$ since this sum has exactly the same form as appeared in the proof of (3). \square

In fact we can explicitly solve the recursion formula of $A_{\mu, \bar{\lambda}, q}$ appeared in the previous proposition. This result is needed in the author's following paper [Taka] and included in Appendix C.

Now Step (A) and (B) have been accomplished.

5.5. Step (C). We multiply $g_{\lambda}^{(k)}$ by $g_P^{(k)}$, and express it as a linear combination of K - k -Schur functions, and solve it:

Theorem 30. *Let P and $\alpha_P(u)$ (for $u \in \mathbb{Z}_{>0}$) be as in (NP) in Section 3, before Proposition 14. Let $\lambda, \bar{\lambda}, \bar{l}$ be as in (N λ) in Section 5.1. Write $r = \lambda_{l(\lambda)}$. Assume $\max_i \{t_i\} < \bar{\lambda}_{\bar{l}}$. Then we have*

$$(1) \quad g_P^{(k)} g_{\lambda}^{(k)} = \sum_{s=0}^r (-1)^s \binom{\alpha_P(r+1-s)}{s} g_{P \cup \bar{\lambda} \cup (r-s)}^{(k)}.$$

$$(2) \quad g_{P \cup \lambda}^{(k)} = g_P^{(k)} \sum_{s=0}^r \binom{\alpha_P(r) + s - 1}{s} g_{\bar{\lambda} \cup (r-s)}^{(k)}.$$

In particular, if $t_n < r$ then $\alpha_P(r) = 0$ and

$$g_{P \cup \lambda}^{(k)} = g_P^{(k)} g_{\lambda}^{(k)}.$$

Proof. (2) follows from (1) and Lemma 21. We prove (1) by induction on $\bar{l} \geq 0$. The case $\bar{l} = 0$ was proved in Proposition 20 and Theorem 23. Assume $\bar{l} \geq 1$.

From Lemma 29,

$$(\text{LHS}) = g_P^{(k)} \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_l}} \sum_q A_{\mu, \bar{\lambda}, q} g_{\mu}^{(k)} \sum_{i \geq 0} \binom{q+i-1}{i} h_{r-|\mu/\bar{\lambda}|-i},$$

by the induction hypothesis, we have $g_P^{(k)} g_{\mu}^{(k)} = g_{P \cup \mu}^{(k)}$ in the above summation. Hence

$$= \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_l}} \sum_q A_{\mu, \bar{\lambda}, q} g_{P \cup \mu}^{(k)} \sum_{i \geq 0} \binom{q+i-1}{i} h_{r-|\mu/\bar{\lambda}|-i},$$

then by Lemma 26(2), (notice that $\mu_{\bar{l}} \geq \bar{\lambda}_{\bar{l}} > \max_i(t_i)$)

$$= \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_t}} \sum_q A_{\mu, \bar{\lambda}, q} \sum_{\substack{\mu \subset \nu \subset R'_t \\ \nu/\mu: \text{ h.s.}}} \sum_{i \geq 0} \binom{q+i-1}{i} T'_{P, \nu, r-|\nu/\bar{\lambda}|-i, r_{\nu, \mu}},$$

then by doing the same argument as Lemma 29(1), (formally replacing T_{\dots} by $T'_{P, \dots}$ and using Lemma 28(2) instead of (1), the proof works)

$$= \sum_{i \geq 0} \binom{r_{\bar{\lambda}\bar{\lambda}} + i - 1}{i} T'_{P, \bar{\lambda}, r-i, r_{\bar{\lambda}\bar{\lambda}}},$$

then by Lemma 28(2),

$$\begin{aligned} &= \sum_{s=0}^r \binom{-\alpha_P(r+1-s) + s - 1}{s} g_{P \cup \bar{\lambda} \cup (r-s)}^{(k)} \\ &= \sum_{s=0}^r (-1)^s \binom{\alpha_P(r+1-s)}{s} g_{P \cup \bar{\lambda} \cup (r-s)}^{(k)}. \end{aligned}$$

□

Now we can achieve our goal in this section.

Theorem 31. For $1 \leq t_1 < \dots < t_m \leq k$ and $a_1, \dots, a_m > 0$,

$$g_{R_{t_1}^{a_1} \cup \dots \cup R_{t_m}^{a_m}}^{(k)} = g_{R_{t_1}^{a_1}}^{(k)} \cdots g_{R_{t_m}^{a_m}}^{(k)}.$$

Proof. Use induction on $m > 0$.

The base case $m = 1$ is obvious. Assume $m > 1$.

Applying Proposition 18 for $\lambda = R_{t_m}^i$ and $t = t_m$, we have

$$g_{R_{t_m}^{i+2}}^{(k)} = g_{R_{t_m}^{i+1}}^{(k)} \frac{g_{R_{t_m} \cup R_{t_m}}^{(k)}}{g_{R_{t_m}}^{(k)}}.$$

Multiplying this for $i = 0, \dots, a_m - 2$, we have

$$(13) \quad g_{R_{t_m}^{a_m}}^{(k)} = g_{R_{t_m}}^{(k)} \left(\frac{g_{R_{t_m} \cup R_{t_m}}^{(k)}}{g_{R_{t_m}}^{(k)}} \right)^{a_m - 1}.$$

Put $P = R_{t_1}^{a_1} \cup \dots \cup R_{t_{m-1}}^{a_{m-1}}$.

Similarly applying Proposition 18 for $\lambda = P \cup R_{t_m}^i$ and $t = t_m$, then multiplying this for $i = 0, \dots, a_m - 2$, we have

$$g_{P \cup R_{t_m}^{a_m}}^{(k)} = g_{P \cup R_{t_m}}^{(k)} \left(\frac{g_{R_{t_m} \cup R_{t_m}}^{(k)}}{g_{R_{t_m}}^{(k)}} \right)^{a_m - 1}.$$

On the other hand, applying the previous theorem for P , $\lambda = R_{t_m}$, we have $g_{P \cup R_{t_m}}^{(k)} = g_P^{(k)} g_{R_{t_m}}^{(k)}$.

Hence we have

$$g_{P \cup R_{t_m}^{a_m}}^{(k)} = g_P^{(k)} g_{R_{t_m}}^{(k)} \left(\frac{g_{R_{t_m} \cup R_{t_m}}^{(k)}}{g_{R_{t_m}}^{(k)}} \right)^{a_m - 1} = g_P^{(k)} g_{R_{t_m}^{a_m}}^{(k)} = g_{R_{t_1}^{a_1}}^{(k)} \cdots g_{R_{t_m}^{a_m}}^{(k)},$$

where the last equality follows by the induction hypothesis. \square

6. DISCUSSIONS

In this section we state some conjectures, that are consistent with our results in previous sections.

Conjecture 32. *For all $\lambda \in \mathcal{P}_k$ and $P = R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m}$, write*

$$g_{P \cup \lambda}^{(k)} = g_P^{(k)} \sum_{\mu} a_{P, \lambda, \mu} g_{\mu}^{(k)}.$$

Then $a_{P, \lambda, \mu} \geq 0$ for any μ .

In the case $P = R_t$, it is observed that $a_{R_t, \lambda, \mu} = 0$ or 1. Moreover, the set of μ such that $a_{R_t, \lambda, \mu} = 1$ is expected to be an “interval”, but we have to consider the *strong order* on $\mathcal{P}_k \simeq \mathcal{C}_{k+1} \simeq \tilde{S}_{k+1}/S_{k+1}$, which can be seen as just inclusion as shapes in the poset of cores, or strong Bruhat order on the affine symmetric group. Namely, the strong order $\lambda \leq \mu$ on \mathcal{P}_k is defined by $\mathfrak{c}(\lambda) \subset \mathfrak{c}(\mu)$. Notice that $\lambda \preceq \mu \implies \lambda \subset \mu \implies \lambda \leq \mu$ for $\lambda, \mu \in \mathcal{P}_k$. Then,

Conjecture 33. *For all $\lambda \in \mathcal{P}_k$ and $1 \leq t \leq k$, there exists $\mu \in \mathcal{P}_k$ such that*

$$g_{R_t \cup \lambda}^{(k)} = g_{R_t}^{(k)} \sum_{\mu \leq \nu \leq \lambda} g_{\nu}^{(k)}.$$

Assuming this conjecture, we shall write $\text{minindex}(\lambda, t) = \mu$.

We can make some conjectures about the behavior of minindex :

- It is expected that if λ gets “bigger” with respect to inclusion, then minindex gets weakly bigger in the strong order. Namely,
For any two elements $\mu \subset \lambda$ of \mathcal{P}_k , we have $\text{minindex}(\mu, t) \leq \text{minindex}(\lambda, t)$.
- If a bounded partition has a form $R_s \cup \lambda$ for $s \neq t$, its minindex still be expected to contains R_s , and the “remaining part” is bigger or equal to $\text{minindex}(\lambda, t)$ in the strong order:
For all $\lambda \in \mathcal{P}_k$ and $1 \leq s \neq t \leq k$, $\text{minindex}(R_s \cup \lambda, t)$ has the form $R_s \cup \mu$ and $\text{minindex}(\lambda, t) \leq \mu$.
- If a bounded partition has a form $R_s \cup R_s \cup \lambda$ for $s \neq t$, its minindex would be equal to the union of R_s and $\text{minindex}(R_s \cup \lambda)$:
For all $\lambda \in \mathcal{P}_k$ and $1 \leq s \neq t \leq k$, we have $R_s \cup \text{minindex}(R_s \cup \lambda, t) = \text{minindex}(R_s \cup R_s \cup \lambda, t)$.

Next, consider a bounded partition that has a form $R_t \cup \lambda$. We wrote

$$\frac{g_{R_t \cup R_t \cup \lambda}^{(k)}}{g_{R_t}^{(k)}} = \sum_{\text{minindex}(R_t \cup \lambda, t) \leq \gamma \leq R_t \cup \lambda} g_{\gamma}^{(k)}.$$

On the other hand, by Proposition 18

$$\frac{g_{R_t \cup R_t \cup \lambda}^{(k)}}{g_{R_t}^{(k)}} = \frac{g_{R_t \cup \lambda}^{(k)}}{g_{R_t}^{(k)}} \frac{g_{R_t \cup R_t}^{(k)}}{g_{R_t}^{(k)}} = \sum_{\text{minindex}(\lambda, t) \leq \mu \leq \lambda} g_{\mu}^{(k)} \sum_{\nu \subset R_t} g_{\nu}^{(k)}.$$

Now we can expect that for any $\mu \in \mathcal{P}_k$,

$$g_\mu^{(k)} \sum_{\nu \subset R_t} g_\nu^{(k)} = \sum_{\gamma \in I_{\mu,t}} g_\gamma^{(k)},$$

where $I_{\mu,t}$ is an order filter of the interval $[\emptyset, R_t \cup \mu]$ (in \mathcal{P}_k with the strong order) such that

$$\bigsqcup_{\mu \in [\text{minindex}(\lambda, t), \lambda]} I_{\mu,t} = [\text{minindex}(R_t \cup \lambda, t), R_t \cup \lambda].$$

APPENDIX A. EXAMPLES

In this section we sometimes abbreviate $\sum_\lambda a_\lambda g_\lambda^{(3)}$ as $\sum_\lambda a_\lambda \lambda$ for ease to see.

TABLE 1. $k = 3$. The table of $g_{Q \cup \lambda}^{(3)} / g_Q^{(3)}$ for $Q = R_{t_1} \cup \dots \cup R_{t_n}$ ($1 \leq t_1 < \dots < t_n \leq k$), $\lambda \subset (1^2 2^1 3^0)$

$Q \backslash \lambda$	\square	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\square \square$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix}$
$\square \square$	$\square + \emptyset$	$\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\square \square + \square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$
$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$	$\square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square + \emptyset$	$\square \square + \square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + \square \square + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + \square + \emptyset$
$\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$	$\square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square + \emptyset$	$\square \square$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \end{smallmatrix}$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + \square \square$
$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix}$	$\square + 2\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square + \emptyset$	$\square \square + 2\square + 3\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + 2\begin{smallmatrix} \square \\ \square \end{smallmatrix} + 2\square + 2\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} + 2\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + 3\begin{smallmatrix} \square \\ \square \end{smallmatrix} + 2\square \square + 3\square + 3\emptyset$
$\begin{smallmatrix} \square \\ \square & \square \\ \square & \square \end{smallmatrix}$	$\square + 2\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square + \emptyset$	$\square \square + \square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} + 2\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \square \square + 2\begin{smallmatrix} \square \\ \square \end{smallmatrix} + 2\square \square + 2\square + 2\emptyset$
$\begin{smallmatrix} \square \\ \square \\ \square & \square \\ \square & \square \end{smallmatrix}$	$\square + 2\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\square + 3\emptyset$	$\square \square + \square + \emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\square \square + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\square + 2\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\square \square + 2\begin{smallmatrix} \square \\ \square \end{smallmatrix} + 2\square \square + 3\square + 3\emptyset$
$\begin{smallmatrix} \square \\ \square \\ \square \\ \square & \square \\ \square & \square \end{smallmatrix}$	$\square + 3\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\square + 3\emptyset$	$\square \square + 2\square + 3\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\square \square + 2\begin{smallmatrix} \square \\ \square \end{smallmatrix} + 4\square + 5\emptyset$	$\begin{smallmatrix} \square & \square \\ \square & \square \\ \square & \square \end{smallmatrix} + 2\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 3\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + 2\square \square + 5\begin{smallmatrix} \square \\ \square \end{smallmatrix} + 5\square \square + 8\square + 9\emptyset$

TABLE 2. $k = 3$. The table of $\text{minindex}(\lambda, t)$ for $|\lambda| \leq 6$.

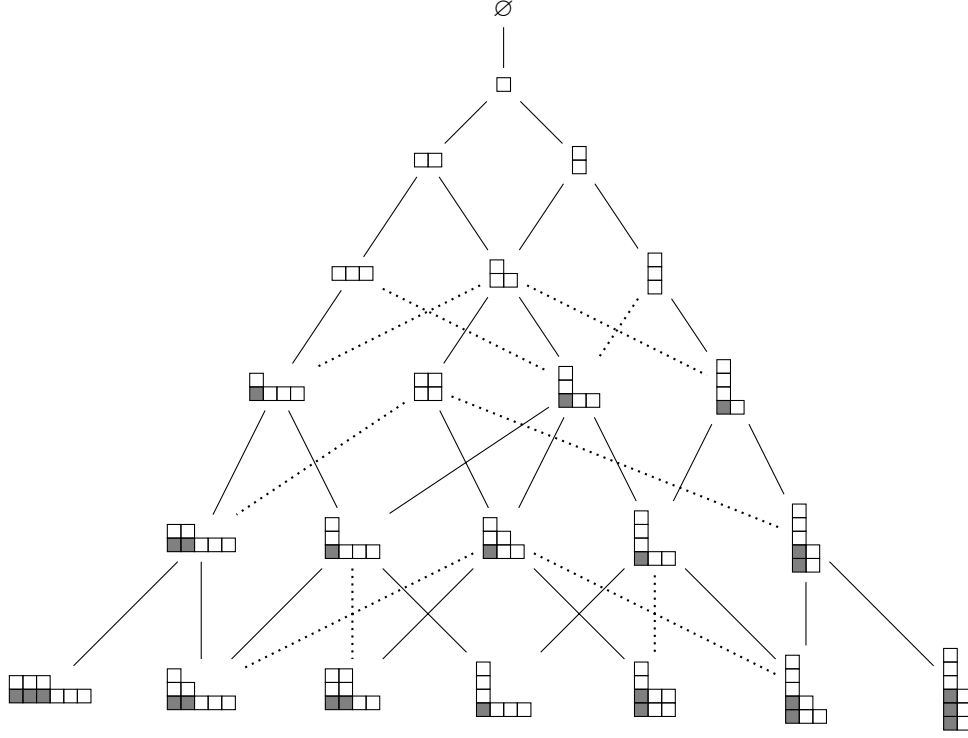


FIGURE 2. The poset of 4-cores (up to those of size 6). The weak cover relations correspond to the solid lines, and the strong cover relations correspond to the solid or dotted lines.

APPENDIX B. PROOF OF PROPOSITION 3

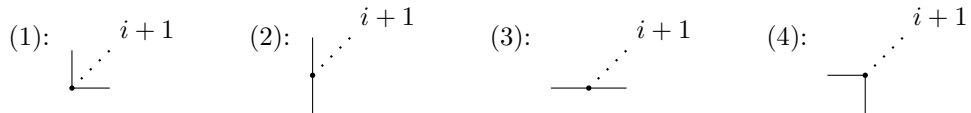
(4) \implies (1): The latter condition $\tau \prec \exists \tau^{(1)} \prec \dots \prec \exists \tau^{(r)} = \kappa$ is obvious.

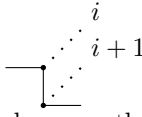
If κ/τ is not a horizontal strip, then $(a, b), (a+1, b) \in \kappa/\tau$ for $\exists a, b$. Write their residues $i = b - a, i-1 = b - (a+1)$. Then a s_{i-1} -action should be performed after a s_i -action.

Then the representation of (4) has the form $\kappa = \dots s_{i-1} \dots s_i \dots \tau$, which contradicts (4).

(1) \implies (4): Assume $\kappa = \dots s_i \dots s_{i+1} \dots \tau$, $|\kappa|_{k+1} = |\tau|_{k+1} + r$, and κ/τ is a horizontal strip.

Consider the moment just before performing the action of s_{i+1} . At that time the situation around each extremal cell of residue $i+1$ is one of the following:


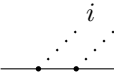


In the case (1), furthermore it should be  since κ/τ is a horizontal

strip. Besides, the case (1) should happen because the action of s_{i+1} must add more than or equal to one box.

In fact the case (4) never happens since the action of s_{i+1} does not remove boxes.

The case (3) is divided to

(3-1):  and (3-2): .

The case (3-1) should happen since later the action of s_i must add more than or equal to one box.

Thus we have a contradiction that there are both addable corners and removable corners of residue i in this moment.

APPENDIX C. EXPLICIT DESCRIPTION OF $A_{\mu, \bar{\lambda}, q}$

Lemma 34. *In the setting of Lemma 29,*

$$A_{\mu, \bar{\lambda}, q} = \begin{cases} (-1)^{|\mu/\bar{\lambda}|} & (\text{if } \mu/\bar{\lambda} : \text{vertical strip and } q = |\mu/\bar{\lambda}| + r_{\mu'\bar{\lambda}'}), \\ 0 & (\text{otherwise}). \end{cases}$$

Proof. We fix $\bar{\lambda}$, and set $f_{\mu}(t) := \sum_q A_{\mu, \bar{\lambda}, q} t^q \in \mathbb{Z}[t]$. Then the definition of $A_{\mu, \bar{\lambda}, q}$ (in the statement of Lemma 29) is transformed into the recursion formula

$$\begin{aligned} f_{\bar{\lambda}}(t) &= t^{r_{\bar{\lambda}}}, \\ f_{\mu}(t) &= - \sum_{\substack{\mu/\kappa: \text{h.s.} \\ \bar{\lambda} \subset \kappa \subsetneq \mu}} t^{r_{\mu\mu} - r_{\mu\kappa}} f_{\kappa}(t) \quad \text{for } \mu \neq \bar{\lambda}, \end{aligned}$$

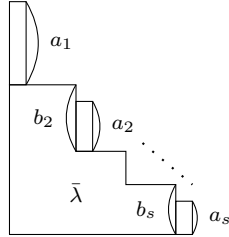
and the desired result becomes the condition

$$f_{\mu}(t) = \begin{cases} (-1)^{|\mu/\bar{\lambda}|} t^{|\mu/\bar{\lambda}| + r_{\mu'\bar{\lambda}'}} & (\text{if } \mu/\bar{\lambda} : \text{vertical strip}) \\ 0 & (\text{otherwise}) \end{cases}.$$

We prove it by induction on $|\mu|$ (for μ satisfying $\bar{\lambda} \subset \mu \subset R'_l$). The base case $\mu = \bar{\lambda}$ is obvious by definition.

Then we assume $\bar{\lambda} \subsetneq \mu \subset R'_l$. First we consider the case where $\mu/\bar{\lambda}$ is a vertical strip. In this case we put

$$\begin{aligned} \{x_1, \dots, x_s\} &:= \{x \mid \bar{\lambda}'_x < \mu'_x\} \quad (x_1 < \dots < x_s), \\ a_i &:= \mu'_{x_i} - \bar{\lambda}'_{x_i}, \\ b_i &:= \bar{\lambda}'_{x_{i-1}} - \bar{\lambda}'_{x_i} (\geq a_i) \quad (\text{if } x_1 = 1 \text{ set } b_1 = \infty), \end{aligned}$$



and we denote by $\kappa(c_1, \dots, c_s)$ the partition defined by

$$\kappa(c_1, \dots, c_s)'_x = \begin{cases} \bar{\lambda}'_x + c_i & \text{if } x = x_i \text{ for some } i \\ \bar{\lambda}'_x & \text{otherwise} \end{cases}$$

for $0 \leq c_i \leq a_i$ ($1 \leq i \leq s$). In particular $\kappa(0, \dots, 0) = \bar{\lambda}$ and $\kappa(a_1, \dots, a_s) = \mu$.

Since $|\kappa(c_1, \dots, c_s)/\bar{\lambda}| = \sum_i c_i$ and $r_{\kappa(c_1, \dots, c_s)'\bar{\lambda}'} = r_{\bar{\lambda}\bar{\lambda}'} - \#\{i \mid c_i = b_i\}$, we have

$$\begin{aligned} f_{\kappa(c_1, \dots, c_s)}(t) &= (-1)^{|\kappa(c_1, \dots, c_s)/\bar{\lambda}|} t^{|\kappa(c_1, \dots, c_s)/\bar{\lambda}| + r_{\kappa(c_1, \dots, c_s)'\bar{\lambda}'}} \\ &= (-1)^{\sum_i c_i} t^{r_{\bar{\lambda}\bar{\lambda}'} + \sum_i (c_i - \delta[c_i = b_i])}, \end{aligned}$$

for $0 \leq c_i \leq a_i$ and $(c_1, \dots, c_s) \neq (a_1, \dots, a_s)$, by the induction hypothesis.

For $S \subset \{1, \dots, s\}$, we set $\kappa(S) = \kappa(a_1 - \delta[1 \in S], \dots, a_s - \delta[s \in S])$. Then

$$\begin{cases} \bar{\lambda} \subset \kappa \subsetneq \mu \\ \mu/\kappa: \text{ horizontal strip} \end{cases} \iff \kappa = \kappa(S) \text{ for } \emptyset \neq S \subset \{1, \dots, s\}.$$

Therefore,

$$\begin{aligned} f_\mu(t) &= - \sum_{\substack{\mu/\kappa: \text{ h.s.} \\ \bar{\lambda} \subset \kappa \subsetneq \mu}} t^{r_{\mu\mu} - r_{\mu\kappa}} f_\kappa(t) \\ &= - \sum_{\emptyset \neq S \subset \{1, 2, \dots, s\}} t^{r_{\mu\mu} - r_{\mu\kappa(S)}} f_{\kappa(S)}(t) \\ &= - t^{r_{\mu\mu} - r_{\mu\kappa(\{1\})}} f_{\kappa(\{1\})}(t) \\ &\quad - \sum_{\emptyset \neq T \subset \{2, \dots, s\}} \underbrace{\left(t^{r_{\mu\mu} - r_{\mu\kappa(T)}} f_{\kappa(T)}(t) + t^{r_{\mu\mu} - r_{\mu\kappa(\{1\} \cup T)}} f_{\kappa(\{1\} \cup T)}(t) \right)}_{(X)}. \end{aligned}$$

In fact it can be proved that $(X) = 0$ by the following Claim 1 and Claim 2.

Claim 1.

$$r_{\mu, \kappa(\{1\} \cup T)} = r_{\mu, \kappa(T)} - \delta[a_1 < b_1] \quad (\text{for all } T \subset \{2, \dots, s\}).$$

Proof of Claim 1: Reduced to next lemma:

Lemma 35. Let $\gamma \subset \beta$ and $y = (r, c)$ be an addable corner of γ . Put $\tilde{\gamma} = \gamma \cup \{y\}$. Assume that $\tilde{\gamma}_1 + l(\tilde{\gamma}) \leq k + 1$ and y is β -nonblocked. Then

$$r_{\beta\tilde{\gamma}} - r_{\beta\gamma} = \begin{cases} 0 & \text{(if } (r, c-1) \text{ is a } \beta\text{-nonblocked removable corner of } \gamma), \\ 1 & \text{(otherwise).} \end{cases}$$

Proof of Lemma 35. Note that $r_{\beta\tilde{\gamma}} = \#\{\beta\text{-nonblocked } \tilde{\gamma}\text{-removable corners}\}$ since $\tilde{\gamma} \subset R'_l$, and the same equality holds for $r_{\beta\gamma}$.

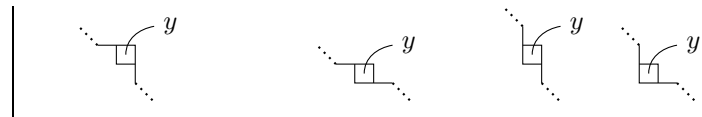
If z is a β -blocked (resp. nonblocked) removable corner of γ other than $(r-1, c)$ or $(r, c-1)$, then z is also β -blocked (resp. nonblocked) removable corner of $\tilde{\gamma}$, and vice versa. Note that

- $y = (r, c)$ is a β -nonblocked removable corner of $\tilde{\gamma}$, and not in γ .
- $(r, c-1)$ is not a removable corner of $\tilde{\gamma}$.
- $(r-1, c)$ is not a removable corner of $\tilde{\gamma}$. Even if $(r-1, c)$ is a removable corner of γ , it is β -blocked.

Hence we conclude

$$r_{\beta, \tilde{\gamma}} - r_{\beta, \gamma} = \begin{cases} 0 & \text{(if } (r, c-1) \text{ is a } \beta\text{-nonblocked removable corner of } \gamma), \\ 1 & \text{(otherwise).} \end{cases}$$

boundary of $\tilde{\gamma}$
around $y = (r, c)$



$r_{\beta \tilde{\gamma}} - r_{\beta \gamma}$

$\delta[(r+1, c-1) \notin \beta]$
 $\delta[(r+1, c-1) \notin \beta]$
 0
 0

□

Claim 2.

$$f_{\kappa(\{1\} \cup T)}(t) = -f_{\kappa(T)}(t) \cdot t^{-\delta[a_1 < b_1]}$$

for $\emptyset \neq T \subset \{2, \dots, s\}$

Proof of Claim 2: Put $a'_i = a_i - \delta[i \in T]$, then

$$\begin{aligned} f_{\kappa(\{1\} \cup T)}(t) &= (-1)^{|\mu/\bar{\lambda}| - |T| - 1} t^{r_{\bar{\lambda}\bar{\lambda}} + (a_1 - 1) + \sum_{i>1} (a'_i - \delta[a'_i = b_i])} \\ &= -(-1)^{|\mu/\bar{\lambda}| - |T|} t^{r_{\bar{\lambda}\bar{\lambda}} + (a_1 - \delta[a_1 = b_1]) - \delta[a_1 < b_1] + \sum_{i>1} (a'_i - \delta[a'_i = b_i])} \\ &= -f_{\kappa(T)}(t) \cdot t^{-\delta[a_1 < b_1]}. \end{aligned}$$

(End of the proof of Claim 2)

Hence,

$$\begin{aligned} f_{\mu}(t) &= -t^{r_{\mu\mu} - r_{\mu\kappa(\{1\})}} f_{\kappa(\{1\})}(t) \\ &= -t^{\delta[a_1 < b_1]} \cdot (-1)^{|\mu/\bar{\lambda}| - 1} t^{r_{\bar{\lambda}\bar{\lambda}} + \sum_i (a_i - \delta[a_i = b_i]) - \delta[a_1 < b_1]} \\ &= (-1)^{|\mu/\bar{\lambda}|} t^{r_{\bar{\lambda}\bar{\lambda}} + \sum_i (a_i - \delta[a_i = b_i])} \\ &= (-1)^{|\mu/\bar{\lambda}|} t^{|\mu/\bar{\lambda}| + r_{\mu'\bar{\lambda}'}}. \end{aligned}$$

This completes the proof in the case where $\mu/\bar{\lambda}$ is a vertical strip.

Next we consider the case where $\mu/\bar{\lambda}$ is *not* a vertical strip.

We take the same x_i, a_i, b_i ($1 \leq i \leq s$) as above (in this case we have $a_i > b_i$ for some i), and $\kappa(c_1, \dots, c_s)$ ($0 \leq c_i \leq a_i, 1 \leq i \leq s$, so long as adding c_i cells on top of the x_i -th column of $\bar{\lambda}$, for all i , yields a Young diagram of a partition) and $\kappa(S)$ ($S \subset \{1, 2, \dots, s\}$ but bound by the same restriction).

Notice that $\kappa(c_1, \dots, c_s)/\bar{\lambda}$ is a vertical strip if and only if $c_i \leq b_i$ for all i .

Now we have

$$f_{\mu}(t) = - \sum_{\substack{\mu/\kappa: \text{ h.s.} \\ \bar{\lambda} \subset \kappa \\ \kappa \neq \mu}} t^{r_{\mu\mu} - r_{\mu\kappa}} f_{\kappa}(t).$$

By the induction hypothesis, we have $f_{\kappa}(t) = 0$ unless $\kappa/\bar{\lambda}$ is a vertical strip. Since μ/κ must be a horizontal strip, κ must have the form $\kappa(S)$. Therefore

$$= - \sum_{\substack{\emptyset \neq S \subset \{1, 2, \dots, s\} \\ a_i - \delta[i \in S] \leq b_i \ (\forall i)}} t^{r_{\mu\mu} - r_{\mu\kappa(S)}} f_{\kappa(S)}(t)$$

If there exists some i such that $a_i > b_i + 1$, then $f_\mu(t) = 0$ since it is equal to an empty sum. So we assume that there is no such i . We set $U := \{i \in \{1, \dots, s\} \mid a_i = b_i + 1\} \neq \emptyset$. It is easily seen that $1 \notin U$. Then

$$\begin{aligned}
&= - \sum_{U \subset S \subset \{1, 2, \dots, s\}} t^{r_{\mu\mu} - r_{\mu\kappa(S)}} f_{\kappa(S)}(t) \\
&= - \sum_{U \subset T \subset \{2, \dots, s\}} \underbrace{\left(t^{r_{\mu\mu} - r_{\mu\kappa(T)}} f_{\kappa(T)}(t) + t^{r_{\mu\mu} - r_{\mu\kappa(\{1\} \cup T)}} f_{\kappa(\{1\} \cup T)}(t) \right)}_{(X)} \\
&= 0 \quad (\text{since } (X) = 0 \text{ by the same reason as the above case}).
\end{aligned}$$

□

Remark. By Lemma 34, Lemma 29(1), say, can be rewritten as:

$$g_\lambda^{(k)} = \sum_{\substack{\mu \text{ s.t.} \\ \bar{\lambda} \subset \mu \subset R'_f \\ \mu/\bar{\lambda}: \text{ v.s.}}} (-1)^{|\mu/\bar{\lambda}|} g_\mu^{(k)} \sum_{i \geq 0} \binom{(|\mu/\bar{\lambda}| + r_{\mu'\bar{\lambda}'} + i - 1)}{i} h_{r - |\mu/\bar{\lambda}| - i}.$$

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